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Abstract

The paper presents a class discussion which was set up within a long term experiment concerning the use of a software, L'Algebrista, to introduce pupils to theoretical thinking and symbolic manipulation. From the analysis of the discussion we will illustrate some aspects of how a teacher can use an instrument of semiotic mediation in order to guide the genesis and evolution of new concepts.

1 Introduction

Within the theoretical framework of Vygotsky's theory Mariotti gave the following characterisation of the idea of using instruments of semiotic mediation:

> "[…] the artefact, acting as a mediator between learners and a teacher, may be used by the teacher to exploit communication strategies aimed at guiding the evolution of meanings within the class community. In other words, the artefact may function as a semiotic mediator.
>
> […] the artefact is exploited by a double use in which it functions as a semiotic mediator. The learner uses the artefact in actions aimed at accomplishing a certain activity, and meanings emerge from this activity; the teacher uses the artefact to direct the learner in the construction of meanings that are mathematically consistent." (Mariotti 2002).

In this paper we are going to discuss how the teacher can use an artefact to direct pupils' construction of mathematical meanings; here the used artefact is a specific algebra software:

> "The computer can also be used as instrument of semiotic mediation. In this case, the teacher uses it to accomplish communication strategies aimed at developing a specific meaning related to the mathematics content, which constitutes the motive of the teaching and learning activity." (Mariotti 2002).

In particular, in the episode we present, the computer is not physically available, but it is evoked by the teacher and by pupils, thus it is anyway present in the specific social activity, a class discussion. Such a situation is typical within the chosen framework and it is coherent with the following hypothesis:

> "Meanings are rooted in the phenomenological experience (actions of the user and feedback of the environment, of which the artefact is a component), but their evolution is achieved by means of social construction in the classroom, under the guidance of the teacher." (Mariotti 2002)

In the following we will analyse some steps passages taken from a class discussion concerning the solution of equations on the basis of the principle "**A=B<=>A–B=0**". Our focus will be on the teacher with particular attention to her interventions aimed at guiding the development of meanings related to the mentioned principle. We will show how her teaching strategy involves and exploits a computer software as an instrument of semiotic mediation.

2 Presentation of the episode

The episode takes place at the beginning of the year with grade 10 pupils. The class is following a long term experimentation concerning the introduction of pupils to theoretical thinking both in algebra (Cerulli 2002, Cerulli et al. 2002) and in geometry (Mariotti 2000, 2002); in both cases the approach is based on the use of specific software, Cabrì in the case of geometry, and L'Algebrista in the case of algebra. For what concerns algebra, symbolic manipulation is interpreted as an activity of proving the equivalence (or non equivalence) of expressions. In particular two expressions are said to be equivalent if it is possible to transform one into the other using the given axioms (ex. distributive, associative and commutative, properties of the operations), or if substituting all the possible numbers to the letters, and computing, we always get the same results. Within such approach, the problem of comparing expressions, results to be at the core of symbolic manipulation, together with the notion of equivalence (Cerulli 2002, Cerulli et al. 2002, Mariotti et al. 2001).

3 The axiom theorem

During the previous year pupils have been following the experimentation for 9 months, and before summer holidays the teacher began to introduce equations starting from the problem of comparing expressions to which pupils were used. The idea is that if we consider expressions that are not equivalent, then we may ask if it is possible to find any number such that if we substitute them to the letters the obtained numerical expressions result to be equivalent.

The teacher (T) begins the lesson by asking pupils to recall what they said 3 months earlier about equations. She knows that they had discussed the statement "**A=B<=> A–B=0**" and she wants to start from this point in order to introduce other principles to be used to cope with equations; in particular she aims at discussing what happens if A and B are not equivalent because she believes that it is a relevant case for equations.

1. T: So, the first question is, do you remember what we have been doing at the end of last year? What did we focus on?
2. Tcl: The axiom theorem *(ita.: assioma teorema)*
3. Cri: axiom theorem one

[…]

6. T: What is it?
7. Tcl: if A is equivalent to B then A minus B is equivalent to zero.
8. T: come to write it *(on the blackboard)* and then explain why we called it axiom theorem

[…]

12. *Tcl writes on the blackboard:* a == b <=> a – b

[…]

14. T: do you remember why did we call it axiom theorem? Is it normal to call something axiom

theorem?

[…]

20. Bzc: we didn't know if…it was proved, we took it as an axiom, last year, but if later we are able to prove it … we left it undecided.

21. T: […] So, we have this axiom theorem, what did it bring to us? What did we do with this axiom theorem? Nothing? Did we just look at it?

Once the "axiom theorem" principle is stated the teacher shifts the focus on its operative aspects ([21]). A key point of our approach is that axioms and theorems are conceived as instruments to transform expressions (Cerulli 2002), thus they have a strong operative meaning. In particular we hypothesise that such a meaning is fostered by creating a correspondence between axioms (and theorems) and the buttons of the used software, L'Algebrista. The teacher here wants to build a meaning of the axiom theorem as instrument to solve equations so she proposes an equation, and pupils actually solve it using such instrument (for a matter of space we omit this part).

4 The "Insert Expression" button

In the first part of the discussion the teacher does not explicitly refer to L'Algebrista which is not available, as the episode takes place in a normal classroom with no computers. The first reference to L'Algebrista occurs while Tcl is solving the given equation on the blackboard:

158. T: so, before using the axiom theorem, and before using the distributive property, we shell use "insert expression"

The button "Insert Expression" (*ita.*: "inserisci espressione") is the first command to be used in L'Algebrista in order to manipulate an expression: by clicking such button the expression is inserted into an environment where it is possible to operate only using the available buttons. Here we hypothesise that the this specific intervention of the teacher has three different aims/effects:

1. there is a parallel between L'Algebrista and the theory pupils are working with, so to "Insert expression" in paper and pencil means to restrict ourselves to use only a specific set of axioms and theorems, the local theory the class is using at the moment of the activity. Here in this specific case it means in particular that it is possible to use the axiom theorem;
2. when the teacher asks pupils to "insert expression" in paper and pencil, it is a kind of request of simulating the behaviour of L'Algebrista which in particular is very rigorous and rigid;
3. the role played by buttons when using L'Algebrista is that of instruments to transform expression, concrete instruments, thus the act of simulating the introduction of a new theorem (the "axiom theorem" here) into such kind of activities may foster its operative meanings.

5 The production of new theorems

From [158] to [236] the focus is on the solution of a specific equations, and we have few references to L'Algebrista, all of them concerning very operative aspects that we do not take into account here. Anyway till now the problem to be solved is a mathematical one (an equation) and the environment where to solve it is typical of mathematics: paper and pencil. Here the role of L'Algebrista is basically to furnish instruments to solve such problem, but what pupils are really using are paper and pencil versions of such instruments, they are not using buttons, neither transforming expressions written on the screen of a computer. In the following part, even if the computer is still absent, there is a change caused by teacher's intervention [237]:

237. T: […] have we ever solved equations within L'algebrista?

238. Chorus: no

239. T: no. So, Michele *(the developer of the software)* is here, we want to tell him what buttons we need in order to solve equations. He will add buttons to L'Algebrista, so what buttons will we require him to add?

240. Fmn: the axiom theorem!

A mathematical concept, the axiom theorem, has been produced by a class activity focused on a mathematical problem, that of equations, and without explicit references to L'Algebrista. Now, such a concept becomes a new element of the software, it becomes a button. Such a change of status of the axiom theorem has some relevant consequences; one of them is strictly related to the very nature, and rigidity, of the software, and leads to the production of a new theorem.

First of all the teacher tries to point out the fact that the syntax used by pupils is not the same as the syntax used by L'Algebrista, where "**a-b**" would be written "**a+(-1)*b**", so she asks:

272. T: ok? But…do you think L'Algebrista would like such button?

273. Cri: no

274. T: why?

275. *confusion*

276. T: if we think of L'Algebrista's mentality …

277. Cri: that from a minus b equal zero I get a equal b it is ok …

278. Tcl: ah! Because we need to do the "insert expression"

279. T: that is … he (*L'Algebrista*) doesn't like so much that "a minus b"…

Tcl gets to what the teacher is aiming and recalls the "insert expression" [278] button that, as a class convention, brings with itself also the specific syntax of L'Algebrista. Here we may notice passage [276] where the teacher not only recalls L'Algebrista, but also its "mentality": there is a specific way of reasoning that is associated to the software, and the teacher tries to direct pupils toward such rationality. In fact Cri seems to be really reasoning as if she was L'Algebrista and feels uneasy:

281. Cri: but, how do we know that, for instance, if a is equal to b then a minus b is equal to zero? If a is equal to b? I mean, in the other direction we can do it because a minus b equal zero then we can do a equal b…but…

282. T: so, did you get Cri's problem? She says "if I have that a minus b is equal to zero, then it is ok"…then she says "if a minus b is equal to zero it is ok to say that a is equivalent to b" but she doesn't agree that if a is equivalent to be then a minus b is equivalent to zero.

283. Cri: No: when you apply it …how can you know that a is equivalent to b?

284. T: and in the other direction how can I know that a minus b is equivalent to zero?

285. *silence*

286. T: so, Cri's problem is very serious, but we must clarify it because it seems not be so clear even for her, in fact she can see it only in one direction and no in the other one…so she says "I want this button from L'Algebrista", right? "But if I apply this button here" … she says "how can I, how can L'Algebrista know that a is equivalent to b so that it can transform it?"

[…]

288. T: how can we tell that to L'Algebrista?

Cri's problem is subtle, when we transform "A=B" into "A-B=0" we take that A and B are equivalent as hypothesis, we can assume it whilst L'Algebrista can't, it is impossible to "tell that to L'Algebrista", thus, if it doesn't know such hypothesis, how can the software apply the axiom theorem? In fact in this case the button works transforming mechanically "**A=B**" into "**A-B=0**", and it is responsibility of the user to discuss what may happen if A is not equivalent to B.

340. T: so, that's right, Cri found a problem and she said "but how can L'Algebrista apply such a button? Because" she says "I have the expressions, I write two expression with an equal sign between them, maybe I invent them and I absolutely don't know if they are equivalent or not, then I ask him (*L'Algebrista*) to apply such button and he brings the second expression at the first member, he puts a plus minus one before it, and tells me that it is equivalent to zero. But this is not true in case the two given expressions are not equivalent; it is not true that this ($a+(-1)b$) is equivalent to zero". But what is the problem we are tackling, Cri? Now are we asking L'Algebrista to transform expressions into equivalent expressions, or are we asking L'Algebrista to solve equations?

341. Cri: solve equations

At this point, after the problem has been raised by Cri, the teacher brings the attention back to the original problem of solving equations. From now on she goes back to talk of axioms and theorems and does not talk of "buttons" any more, she does not explicitly refer to L'Algebrista any more.

342. T: to solve equations. Thus, maybe I can ask, I mean my problem is to question for what values of the letter those two numerical expressions are equivalent, thus I will have some values of

the letter for which the numerical expressions are equivalent, thus it is ok to do such a passage; and I will have some values for which I do not know. Now, the important thing becomes another one, if they are not (*equivalent)*, how will their difference be?

343. Cri: different from zero

344. T: right, it will be different from zero

345. T: but also in that case, we shell take this as an axiom otherwise what do we have? A monster?

346. T: so, delete everything *(from the blackboard)*, keep….and prove, with double arrow,…let's see if we are able to prove this theorem that if a is not equivalent to b then, double arrow, a minus b is not equivalent to zero.

347. *Tcl writes*

348. If a =/= b <=> a-b =/= 0

From the discussion of the case of "A not equivalent to B", which was originated by the introduction of the axiom theorem in l'Algebrista, a new theorem is created and it is proved (we omit the proof), using also the axiom theorem. After this new theorem is proved, a new button is created, and it takes the status of "theorem button" (in L'Algebrista we distinguish between buttons that we take as axioms or as theorems):

481. T: now, as we have the theorem, then we have a new button that will not be an axiom button, but will be a theorem button […]

In the rest of the episode the teacher asks pupils to invent new theorems/buttons, in order to use them as instruments to solve equations.

492. T: […] there must be some other theorem for equations, not only those of these two buttons, maybe you can find out others, I mean, it may be the case that if a is equivalent to b, then there is something more apart from the fact that the difference is zero, right? And these new buttons (*the ones pupils are being asked to invent*) maybe be useful to solve equations, so I am asking you, as a homework is […] to think of using some new buttons to solve equations.

Finally we observe that here the evolution of the axiom theorem has not reached an end yet, in fact along the course of following activities, the theorem "**a =/= b <=> a-b =/= 0**" merged into the axiom as a unique instrument to be used to solve equations.

6 Theorems, as buttons, can be used

Teacher's intervention [237], as we said, aims at giving the axiom theorem the status of a button, this has an immediate implication, because as a button it can be used, it is now officially an instrument that pupils can use in their future activities; this is explicitly stressed by the teacher:

409. T: so we have our axiom theorem, no one can private us of it, and we even made a button for it, there it is (*points to a writing on the blackboard representing the button of the axiom theorem*), thus we can use it […]

A given theorem in mathematics is not just a "true statement" but it is also an instrument that can be used to prove other theorems or solve problems. On the other hand, a button can actually be physically used as an instrument to accomplish a certain goal. Here the teacher's intervention aims at exploiting such a parallelism between button and theorem in order to foster the operative meanings of latter. A generic idea of correspondence between buttons and theorems (or axioms) had already been built during the first year of experimentation, thus it can be exploited this specific case.

7 The interplay between two worlds

The episode we showed presents a cycle of production/evolution of concepts that were originated by a voluntary (directed by the teacher) interplay between two worlds: the world of mathematics within L'Algebrista, and the world of mathematics outside L'Algebrista. An exhaustive discussion on the nature of these two words would be too long for the nature of this paper, here we just stress a few key points. First of all, as a notation, we are going to call "***class mathematics world***" the world outside L'Algebrista, so any activity, action, speech, sign, related to this world concerns mathematics and does not contain references to such software. On the other side we will call "***L'Algebrista world***" that of the software, and any activity, action, speech, sign, containing explicit (or implicit) references to the software may be classified as related to such world. Of course it might not always be possible to establish when a given event belongs to a world or to the other, but such distinction will help us analysing how the teacher directed, the evolution/production of some concepts as for instance that of the "axiom theorem".

Fig. 1 presents a scheme of what happened in the episode. During the first part of the class discussion, the axiom theorem was recalled, and was related to the solution of equations. During this first part there is no evidence of any reference to L'Algebrista, thus we placed it on the side of *class mathematics world*. Teacher's intervention [237] explicitly shifts the focus on *L'Algebrista world* and results in the production of a new button, corresponding to the axiom theorem. One of the consequences of such shift is the problem raised by Cri [273-289], she is wondering what happens if A and B are not equivalent, she is wondering how L'Algebrista is able to do it; the button, due to the nature of the software, cannot correspond perfectly with the theorem originated in mathematics, at least in the way it is used. At this point the teacher takes this problem, originated within *L'Algebrista world*, and brings it into *class mathematics world* [340,342], in order to exploit it to produce a new theorem. Such new theorem is discussed, proved, and finally brought into L'Algebrista as a new button [481].

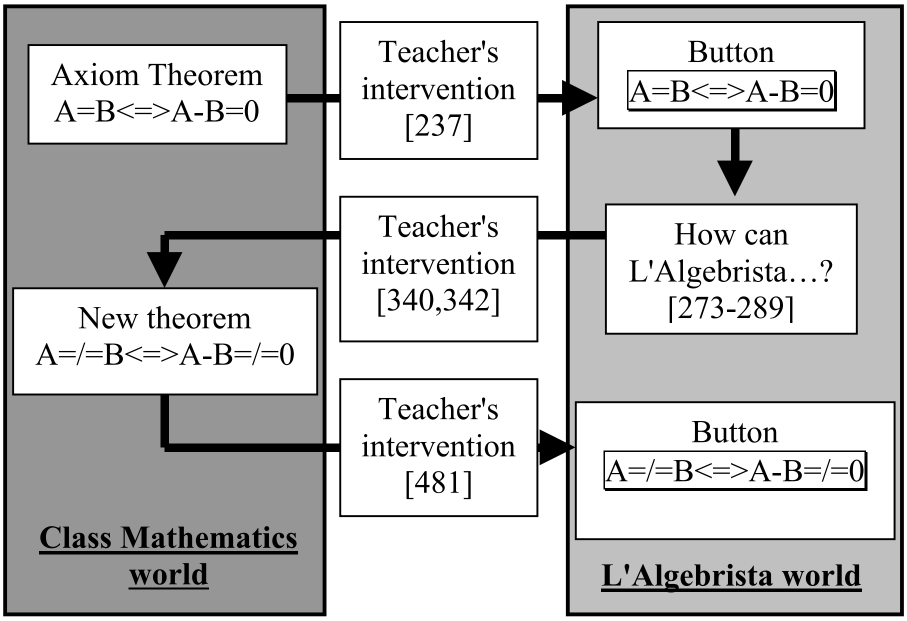

Fig. 1 The interplay between two worlds

The interventions that we just resumed consist mainly in forcing movements back and forth between the two worlds, but in the discussion we individuated another important kind of intervention, like [409] when a correspondence between axioms (and theorems) and buttons is recalled. Such parallelisms is an idea that does not belong to *L'Algebrista world* neither to *class mathematics world*, but it concerns both of them and we may classify it as belonging to some kind of ***linking knowledge*** concerning the relationship between the two worlds. Such relationship is not a perfect correspondence, and to "know" it means to "know" parallelisms between the two worlds but also differences.

In the case of intervention [409] a parallelism is exploited in order to bring operative meanings from L'Algebrista to class mathematics, while in the case of Cri's uneasiness [273-289] the key point is a difference between the two worlds. In fact, if on one hand, in paper and pencil we may treat two given expressions as if they are equivalent, even if they cannot be transformed one into the other; on the other hand it is impossible to tell L'Algebrista to consider equivalent two expressions that cannot be transformed one into the other. During the episode the software is not available, thus Cri's reasoning can be based only on the fact that she knows this specific difference between what she can do inside or outside L'Algebrista.

In conclusion when the teacher moves from one world to the other, she may exploit not just each single world as separated, but also the relationship between them, in order to guide the development of mathematical meanings as separated from meanings concerning the used artefact. For instance, when the teacher [158], asks pupils to "insert expression", she is exploiting also a difference between the two worlds: pupils know that L'Algebrista has its own syntax and that it is different from the one they use normally, *thus* they shift to such syntax after the teacher's intervention; we may suppose that in case if pupils considered L'Algebrista's syntax to be the same as mathematical one, maybe they would not have changed syntax when asked to insert expression.

8 Conclusions

The paper presented a class discussion which was analysed in order to point out how the teacher used an artefact, as an instrument of semiotic mediation, exploiting "communication strategies aimed at guiding the evolution of meanings within the class community" (Mariotti 2002). The interventions we individuated are based on the distinction between two worlds, that of L'Algebrista and that of class mathematics, and on the relationships between them. The teacher exploited her and pupils' knowledge concerning these two worlds and a knowledge concerning their relationships, the second being a *linking knowledge* that may be originated within a hybrid space which is not L'Algebrista world neither class mathematics world. Such a space is characterised by signs and languages, that make it possible to talk about both the two worlds and to compare them. Movements from *L'Algebrista* to *class mathematics* (and vice versa) are possible only if participants to the activity are conscious that there is a distinction between them, and Cri's example shows how such movements can be effective when pupils are acquainted with such distinction. As a consequence we hypothesise that one key point of an effective use of an artefact as instrument of semiotic mediation is to set up activities with focus on the relationship between the *world of the artefact* and the *world of the mathematics we want to teach*. The teacher here plays a key role in individuating and stressing differences and parallelisms between the two worlds and can exploit them in order to guide the construction and evolution of mathematical meanings, as showed in the example of the axiom theorem.

The idea of considering two worlds, *L'Algebrista* and *class mathematics*, together with a *linking knowledge*, is to be intended as a proposal that needs to be refined and defined consistently with the chosen framework. Here we just meant to show how it can help us in describing some key teacher's interventions that are at the core of the use of instruments of semiotic mediation as a paradigm of teaching activity.

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