Introducing pupils to theoretical thinking: the case of algebra

Michele Cerulli

Proceedings of CERME 2, European Research in Mathematics Education II, vol. 1, pg. 139-151. Jarmila Novotna (Eds.), Prague 2002. ISBN 80-7290-075-7.

INTRODUCING PUPILS TO THEORETICAL THINKNG: THE CASE OF ALGEBRA

Michele Cerulli
Dipartimento di Matematica, Università di Pisa

Abstract

Within the theoretical framework of Vygotsky's theory, the paper presents a teaching experiment concerning the introduction of pupils to algebraic theoretical thinking. Starting from the results of a previous study project concerning the use of Cabrì Gèométre to introduce pupils to geometry theory, the experiment is based on the use a algebra microworld "L'Algebrista". Outlines of the classroom experimentation are followed by the analysis of some protocols, according the Vygotskian theory of semiotic mediation.

1 Introduction

As clearly shown by previous research studies the evolution of algebraic symbolism can be described in "procedural-structural" terms (Sfard, 1991).

The procedural character of pupils' conceptions related to literal terms and expressions tends to persist; at the same time, although symbolic manipulations of literal expressions is largely present in school practice, the absence of "structural conceptions" appears evident (Kieran, 1992, p. 397).

In the Italian school, pupils begin fairly soon to be trained in simplifying expressions (first numerical and then literal) and this training is intensively practiced at grade 9, when the first months of the school year are devoted to pupils introduction to 'Algebra'.

Limits related to a procedural approach to symbolic manipulation have been often pointed out, so as the need of a "structural-relational" approach in order to master symbolic manipulation in a productive way (Arzarello, 1991).

Poor strategic decisions has been described, made by students with extensive algebra experience, but unable to identify the right transformation to be accomplished: when the task does not explicitly indicate what transformation has to be performed, pupils are unable to take a decision, "go around in circles" (Kieran, 1992, p. 397) carrying out transformations without any clear goal.

A key point of structural approach is the notion of "equivalence relation" between expressions. Actually, expressions manipulation means substituting an expression with another one which is equivalent. The meaning of the words "*expression"* and "*equivalent"* are not univocally, and a priori, determined, but it shall be so, once a set of axioms is accepted. We consider *"symbolic manipulation"* as characterised by activities of transformation of expressions using the rules given by the assumed axioms and definitions. Thus, symbolic manipulation makes sense within a theoretic system. Certainly this perspective is not very common in school practice (at least in Italy), yet it is exactly the perspective we assumed.

A previous study project, concerning pupils' introduction to geometry theory (Mariotti et al., 1997, 2000), clearly showed how a computer environment may offer

Cerulli, M. (2002). *Introducing pupils to theoretical thinking: the case of algebra*. Proceedings of CERME 2, European Research in Mathematics Education II, vol. 1, pg. 139-151. Jarmila Novotna (Eds.), Prague 2002. ISBN 80-7290-075-7.

a support to overcome the well known difficulties related to theoretical perspective. In particular Mariotti analysed the semiotic mediation that can be accomplished by the teacher using specific instruments offered by the Cabri-géomètre environment (Mariotti, in press).

In the same stream a research project, still in progress, has been set up; a computer microworld, L'Algebrista (Cerulli 1999, Cerulli et al. 2000), was designed incorporating the axioms defining the algebraic equivalence relation. A prototype was realised and experimented in ninth grade classes.

Our basic hypothesis is that algebra's axioms, definitions and theorems, are the main elements involved in the transformation of expressions.

In L'Algebrista *expressions* on the screen can be manipulated using *buttons*. Such computational objects may be interpreted as signs referring to *expressions* and *axioms* (or theorems) within algebra theory; the manipulation of such signs corresponds to symbolic manipulation. In other terms, in the microworld a physical counterpart of expressions and axioms allows the user to visualise and make explicit the mathematical entities end relationships which are involved in symbolic manipulation.

2 Outlines of the classroom experimentation

A first experimentation was, carried out during the school year 1998/1999 in a 9th grade class (Cerulli 1999), and permitted to realise a second version of the prototype, which has been experimented in another class at the same level during the school year 1999/2000. The second teaching experiment represents a junction point between our research concerning algebra and the already mentioned study project, concerning pupils introduction to geometry theory (Mariotti et al., 1997, 2000). The idea is to introduce pupils to theoretical thinking at the same time in geometry and algebra with the support of the environments offered by Cabrì and L'Algebrista. A research project on the effectiveness of the joint use of such microworlds has been planned for the next the school year (2000/2001) in 9th and 10th level classes.

A detailed description of the study project is behind the scopes of this paper, here we just indicate the basic ideas inspiring the sequence of activities concerning algebra.

First of all we recall that our educational goal concerns:

- to introduce pupils to symbolic manipulation;
- to introduce pupils to a theoretical perspective.

According to our hypothesis, the concept of *equivalence relationship* is the basic principle underling symbolic manipulation, thus it represents the starting point of pupils activities.

We introduce the problem of comparing expressions, taking into account the fact that, at this school level, pupils consider numerical expressions as equivalent when they give the same number as result. Thus it is not difficult to negotiate the interpretation of numerical expressions as computation schemes, which will be equivalent if they give the same result.

The idea of interpreting expressions as computation schemes allows one to introduce the properties of sum and multiplication as principles (theory axioms) that determine "a priori" whether two computation schemes lead to the same result: if two expressions are equivalent on the base of such properties then the computation of the two expressions must lead to the same number. Thus a new equivalence relationship between expressions, based on the iterated use of the buttons, is introduced:

if one expression can be transformed into another using the properties of sum and multiplication (our axioms), then the two expressions are equivalent.

In the microworld L'Algebrista this corresponds to:

two expressions are equivalent if it is possible to transform one into the other using the given buttons (representing our axioms).

Once this equivalence relationship is accepted, pupils are asked to compare expressions. A new terminology is introduced: one says that the equivalence of two expressions is **proved** if one expression is transformed into the other using the axioms; vice versa one says that the equivalence is **verified** if the calculation of both the expressions leads to the same result. With literal expressions the difference between *proof* and *verification* becomes even more definite: the use of axioms becomes the only way to state the equivalence between two expressions, whilst numerical verification (substituting the letters with numbers and computing the expressions) becomes the main way to prove that two expressions are not equivalent.
Il Teorematore (Cerulli et al. 2000) can then be used to add a selected choice of proven equivalencies to the set of buttons to be used for new proofs.

3 Semiotic mediation

Within the Vygotskyan framework of semiotic mediation theory, a central role is played by the signs used to mediate mathematical meanings. L'Algebrista was designed as a microworld which could mediate the idea of theory in algebra, and the process of theory building. Algebra theory, as far as imbedded in the microworld, is evoked by the expressions and the commands available in L'Algebrista. According to the Vygotskian theory (Mariotti, in press), expressions and commands may be thought as external signs of the Algebraic theory, and as such, they may become instruments of semiotic mediation (Vygotsky, 1978).
The process of building a theory, by proving, accepting and using new theorems, can be evoked by specific activities within L'Algebrista. Proving that two expressions are equivalent, in algebra, corresponds to proving a theorem, thus in the microworld, *transforming an expression into another,* using the available buttons, corresponds to *proving a theorem*. Furthermore, creating using Il Teorematore, a button corresponding two a new *equivalence relationship*, and adding it to the collection of the available buttons, corresponds to accepting a new theorem. Finally *using a button* created with Il Teorematore corresponds to *using a new theorem*.
In summary, the main instruments of semiotic mediation, offered by L'Algebrista, and related to the theoretic aspects of algebra, are:

- *expressions in L'Algebrista* are signs of *algebraic expressions*;
- *given buttons* are signs of *axioms* and *definitions*;
- *transforming an expression* into an other using the buttons corresponds to *proving* that the two expressions are equivalent, the produced *chain of justified steps* (the justification of each step is reported on its left) corresponds to a *proof*;

Cerulli, M. (2002). *Introducing pupils to theoretical thinking: the case of algebra*. Proceedings of CERME 2, European Research in Mathematics Education II, vol. 1, pg. 139-151. Jarmila Novotna (Eds.), Prague 2002. ISBN 80-7290-075-7.

- *new buttons*, built using Il Teorematore, are signs of *theorems*;
- *adding new buttons* to the set of available buttons is a sign of the meta-theoric operation of *adding new theorems* to a theory.

Some comments, on how such instruments can function as semiotic mediators, are included in the analysis of some protocols.

3.1 Signs derived from L'Algebrista

The representation of an expression in L'Algebrista incorporates its mathematical tree structure, and this structure becomes explicit, "tangible", thanks to the *selection function*, when the user interacts with the environment.

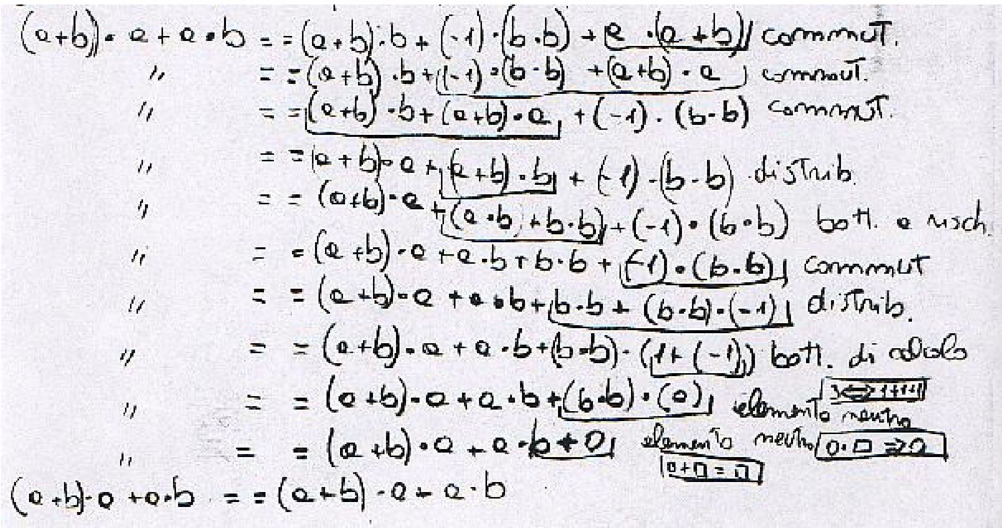


Fig. 1 In the case of a comparison task, performed in paper and pencil environment, the protocol shows that pupils use signs clearly derived from L'Algebrista. In particular the *selection function*, or the iconography of the buttons.

In the case of a comparison task, an example of how pupils may use the *selection function* as an external sign of control of the algebraic structure of an expressions, is provided by the protocol in Fig.1: Lia (9^{th} grade) tries to prove that the two expressions are equivalent, and at each step she underlines (*selects*) a sub-expressions and transforms it using an axiom that applyes. This behaviour recalls the interaction between the user and L'Algebrista: when transforming an expression, one first has to select a sub-expression and then to click on a button representing an axiom that applies. Furthermore, in the example, Lia refers clearly to the buttons of L'Algebrista using the word *button* (Ita.: *bottone*) and reproducing the iconography of the *buttons of neutral elements* (Ita.: *elementi neutro*) and of the *computation buttons* (It.: *bottoni di calcolo*) that she is using. In particular she refers to the following buttons:

- 0+A ➔ A: this button transforms an expression of the kind "0+A" into the expression "A", where "A" can be any expression. This button corresponds to the axiom defining the neutral element of the sum operator.
- 0*A ➔ 0: this button transforms and expression of the kind "0*A" into the expression "0". This corresponds to one of the properties of the "zero" element concerning the multiplication operator; such a property, in our experiment, is assumed to be an axiom.
- 3 ⇔ 1+1+1: converts a number into its decomposition as a sum of *ones*, and, if applied on a sum of numbers, transforms it computing its result. This button

Cerulli, M. (2002). *Introducing pupils to theoretical thinking: the case of algebra*. Proceedings of CERME 2, European Research in Mathematics Education II, vol. 1, pg. 139-151. Jarmila Novotna (Eds.), Prague 2002. ISBN 80-7290-075-7.

corresponds to the definition of sum between numbers, it doesn't apply on letters.

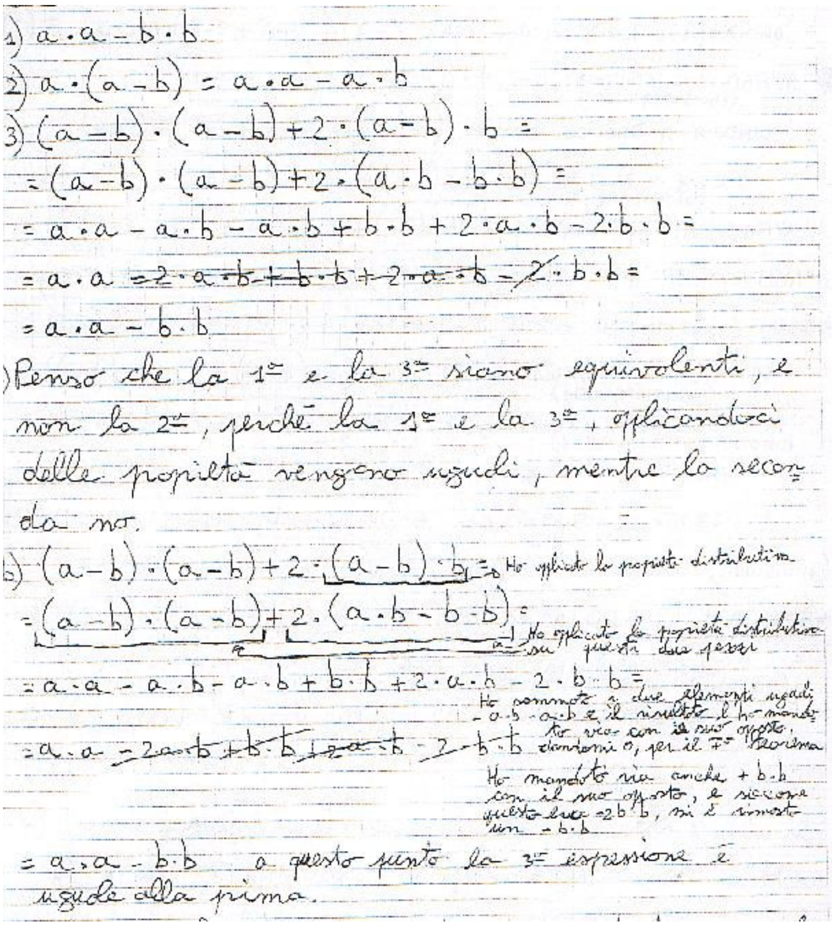
3.2 Making conjectures and proving

Let's consider the following problem given in class, with no computers:

1) Considera le seguenti espressioni: **a*a-b*b** **a*(a-b)** **(a-b)*(a+b)**	1) Consider the following expressions:
a) Quali di esse pensi che siano equivalenti? E quali pensi che non lo siano? Perché? Sapresti dimostrarlo?	a) Which of them do you think are equivalent? Which do you think are not? Why? Can you prove it?
b) Analizza la dimostrazione che hai fatto nel punto precedente ed indica per ogni passaggio fatto se hai utilizzato un teorema o un assioma.	b) Analyse your proof and specify, for each step, if you used a theorem or an axiom.

Pupils are asked to compare three expressions and to find out which of them are equivalent; the produced conjectures are required to be proved.

Fig. 2 Silvio first of all checks the equivalence using his computing skills, once he made his conjecture he uses the properties of the operations (i.e. axioms) and a theorem to prove it. A translation of each statement is reported on the right of the image.



I think the 1st and the 3rd are equivalent, but not the 2nd, because applying the properties they become equal, while the 2nd does not.

I applied the distributive property.

I applied the distributive property on these two pieces.

I summed the two equal terms **–a*b –a*b** and I cancelled its result with it opposite obtaining **0** for the 1st theorem.

I cancelled also **+b*b** with its opposite and as it was **–2b*b** I obtained **–b*b.**

At this point the 3rd expression is equal to the 1st expression

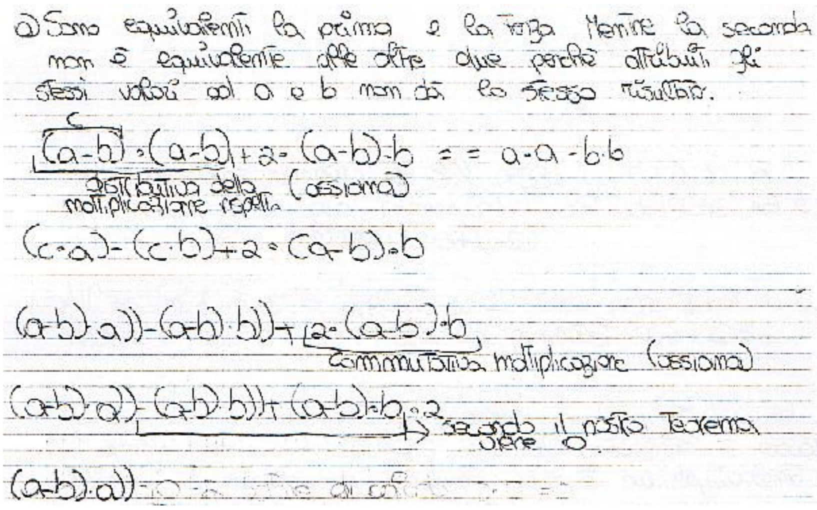
Silvio (Fig. 2) begins reducing the second and the third expression in a form that makes easier comparing them with the first. This part of the protocol looks like typical protocols produced by pupils when asked to *compute* (ita.: "calcolare") expressions. In this case Silvio is not required to compute expressions, but he uses his

Cerulli, M. (2002). *Introducing pupils to theoretical thinking: the case of algebra*. Proceedings of CERME 2, European Research in Mathematics Education II, vol. 1, pg. 139-151. Jarmila Novotna (Eds.), Prague 2002. ISBN 80-7290-075-7.

computing skills to produce a conjecture: as a result he finds out that the third expression is equivalent to the first. Note that Silvio's explanation of how he produced his conjecture anticipates its proof; the properties of the operation, the *axioms* previously introduced, are used by the pupil already during the heuristic phase as *tools* to accomplish the specific task.

In the last part of the protocol, as required, Silvio writes a correct proof of the equivalence of the two expressions referring to axioms and theorems. In particular he refers to the "1st theorem" that the pupils proved on their own, such theorem states that "**a-a=0".** From a formal point of view the chain of equivalent expressions of the second part of the protocol represent a real proof, while the chain reported in the first part does not because steps are not explained referring to algebra theory.

Fig. 3 Marta substitutes letters with numbers to find out which of the three expressions are equivalent; nevertheless she uses axioms and theorems (as she remarks) to prove the equivalence of the first and the third expression.


a) Sono equivalenti la prima e la terza mentre la seconda non è equivalente alle altre due perché attribuiti gli stessi valori ad a e b non dà lo stesso risultato.

(a-b)·(a-b)+2·(a-b)·b = = a·a - b·b

distributiva della moltiplicazione rispetto (assioma)

(c·a)-(c·b)+2·(a-b)·b

(a-b)·a))-(a-b)·b))+2·(a-b)·b

commutativa moltiplicazione (assioma)

(a-b)·a))-(a-b)·b))+(a-b)·b·2

secondo il nostro Teorema viene 0

(a-b)·a))

The first and the third expressions are equivalent, while the second is not because giving the same numerical numbers to **a** and **b** the result is not the same of the other two.

Distributivity of multiplication (axiom).

Commutativity of multiplication (axiom).

Following our theorem this is **0**.

Differently from Silvio, Marta (Fig. 3), doesn't use the properties of the operations to produce her conjecture: she substitutes numbers to letters and computes the obtained expressions. Nevertheless, when proving the equivalence between the first and the third expression, Marta produces a correct formal proof. She reports, at each step, the axiom or theorem she is using and underlines the sub-expression to which each specific axiom/theorem is applied.

In particular she correctly specifies (as required) whether any equivalence relationship is an axiom or a theorem. This distinction corresponds, in our teaching experiment, to the distinction between given principles (axioms) and relationships that were discovered and proved by the students (theorems); it finds its counterpart in L'Algebrista: axioms are represented by given buttons, theorems are produced by pupils with Il Teorematore.

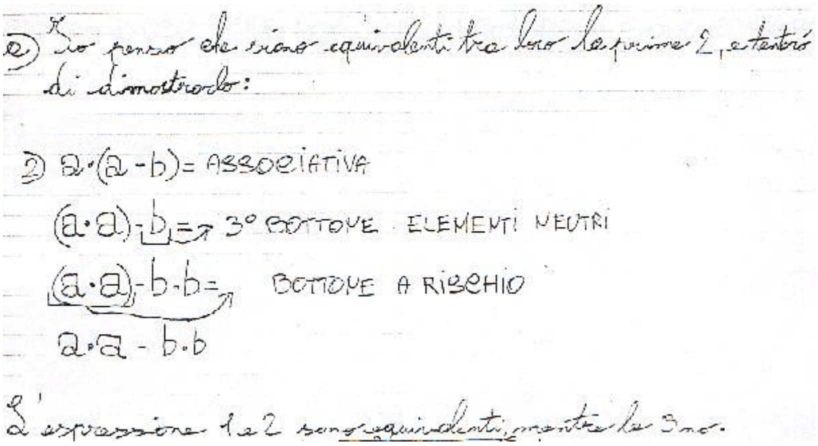
Finally, the fact that Marta uses the words "our theorem", referring to the "1st theorem" mentioned by Silvio, shows how she is conscious that she is using a theorem she produced together with the other pupils.

The last example we consider is the case of Marco (Fig. 4); he doesn't give any explanation of how he produced his conjecture and doesn't seem to be sure of what

Cerulli, M. (2002). *Introducing pupils to theoretical thinking: the case of algebra*. Proceedings of CERME 2, European Research in Mathematics Education II, vol. 1, pg. 139-151. Jarmila Novotna (Eds.), Prague 2002. ISBN 80-7290-075-7.

he found out: he says that he “thinks that” … and he is going “to try to prove” the equivalence between the second and the first expression. What he does is to transform the second expression into the first one referring to the properties of the operation and to the buttons of L’Algebrista. Although he doesn’t produce a correct proof, as the two expressions are not equivalent and he doesn’t use correctly the axioms he mentions, Marco has taken a theoretical perspective: he is conscious that he has to produce a proof and he tries to base his reasoning on the given axioms and theorems represented by the buttons of L’Algebrista.

Fig. 4 Marco tries to prove a wrong conjecture arriving to a wrong conclusion. It is notable how he is conscious that he is trying to produce a proof and how he tries to base his reasoning on the given axioms and theorems represented by the buttons of L’Algebrista.



I think that the first two expressions are equivalent, and I am going **to try** to prove it:

Associative property.

3rd button of neutral elements.
Risky button.

Expressions 1 and 2 are equivalent, while number 3 is not.

4 Conclusions

The development of information technologies raised many issues, one of those concerns the revision of school curricula taking into account the changes brought by this development. The ideas we presented in this paper give an example of a new way to approach symbolic manipulation (Ita. "calcolo letterale"). Our proposal is to be considered in the broader perspective of the introduction of pupils to theoretical thinking. Thus symbolic manipulation has been interpreted taking a theoretical perspective and the particular software environment has been designed as embedding Algebra Theory.

The axioms incorporated in the buttons of L'Algebrista become tools that pupils can learn to use to transform expressions in order to attain activities' goals, and as such they can function as semiotic mediators. The distinction between buttons representing axioms, and buttons for computations, helps distinguishing the terms "proof" and "verification"; and may contribute to build the meaning of *proof* as well as the idea of *theory*. Furthermore the possibility of creating new theorems and making them usable, offered by “Il Teorematore”, lets the student take part in the activity of *theory* evolution.

The presented protocols highlighted how some features of L’Algebrista can mediate some specific concepts related to algebra. In particular it is worth to observe that in the presented examples a central role, seems to be played by the particular set of activities: pupils refer explicitly to the history of the construction of their *theory* by using expressions like "*our theorem*" or "*1st theorem*".

Thus the following questions rise: what kind of activities may a teacher set up to exploit a tool to facilitate processes of semiotic mediation? Which of such processes may happen merely by using the specific tool? And which of them may happen and be effective only thanks to its integration in social interaction with peculiar activities? A research project was set up in order to study such questions. In particular the triangle Teacher-Microworld-Pupils will be studied in terms of semiotic mediation in the case of the joint use of L'Algebrista and Cabrì-Géomètre.

References

Arzarello F.: 1991, *Pre-algebraic Problem Solving*. In J.P.Ponte J.F.Matos, J.M. Matos, D.Fernades, Mathematical Problem Solving and new Information Technologies, Springer.

Arzarello F., Bazzini L., Chiappini P.: 1994, *L'Algebra come strumento di pensiero*. Progetto strategico del C.N.R. "Tecnologie e innovazioni didattiche (innovazioni didattiche per la Matematica)", Quaderno n. 6,

Cerulli M.: 1999, *Uso di manipolatori simbolici nell'introduzione all'algebra: realizzazione e sperimentazione di un prototipo*. Unpublished manuscript, Tesi di Laurea in Matematica, "Università degli Studi di Pisa", Pisa.

Cerulli M. & Mariotti, M.A.: 2000, *A symbolic manipulator to introduce pupils to algebra theory*. Proceedings of *Workshop W6 "Learning Algebra with the Computer a Transdisciplinary Workshop"*. ITS'2000, Montreal.

Kyeran, C.: 1981, *Concepts associated with the equality symbol*. "Educational Studies in Mathematics", 12. D. Reidel Publishing Co., Dordrecht,.

Kyeran, C.: 1992, *The learning and teaching of School Algebra*. In Handbook of Research on Mathematics Teaching and Learning. D. A. Grouws ed., N.C.T.M..

Mariotti M.A.: 2000, Constructions en Geometrie et le probleme de la preuve, *Actes de La X°Ecole d'été de Didactique de Mathematiques*, Houlgate

Mariotti M.A. In press: *Influence of technologies advances on students' mathematical learning*, English L., Tirosh, D.& Bartolini Bussi M. (eds.) Handbook of international research in mathematics education", LEA.

Mariotti M.A., Bartolini Bussi M., Boero P., Ferri F., Garuti R.: 1997, Approaching geometry theorems in contexts: from history and epistemology to cognition, *Proceedings of the 21st PME Conference,* Lathi, pp. I- 180-95.

Mariotti, M.A. & Bartolini Bussi M.G.:1998, From drawing to construction: teachers mediation within the Cabrì environment, in *Proceedings of the 22nd PME Conference,* Stellenbosh, pp. I-180-95.

Noss R. & Hoyles C.: 1997, *Windows on mathematical meanings (learning cultures and computers)*. Kluwer Academic Publishers, Dordrecht,

Prodi G.:1975 *Matematica come scoperta, 1,* D'Anna ed. ,Messina-Firenze,.

Sfard A.: 1991, *On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin*. Educational Studies in Mathematics, 22 (pg. 1-36).

Sfard A.: 1999, *Symbolising mathematical reality into being: How mathematical discourse and mathematical objects create each other*. In P. Cobb, K. E. Yackel, & K. McClain (Eds.), Symbolising and communicating: Perspectives on Mathematical Discourse, Tools, and Instructional Design (pp. 37-98). Mahwah, NJ: Erlbaum.

Vygotsky L.S.: 1987, *Il pensiero cognitivo*. Boringhieri, Torino.