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# Students conceptions: an introduction to a formal characterization

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**Abstract:** We investigate in this paper the complexity of modeling students knowing of mathematics under the constraints of acknowledging both their possible lack of coherency and their local efficiency. For this purpose we propose a formalization of the notion of “conception” as a possible tool to answer the epistemological problem we identify. We apply then this approach to the study of the possible conceptions of “function”, from an historical and then an epistemic point of view. We report the result of a case study in order to illustrate the benefit we expect from this approach. The notions of “conception”, “knowing” and “concept” are then related the one to the other within the model presented.

## 1. FROM BEHAVIOR TO MEANING

The only indicators we have of the good or of the bad functioning of teaching are students’ behaviors and productions, which are consequences of the *knowing*<sup>1</sup> they have constructed and of their relationships to the content taught. But such evaluations are possible and their results are significant only in the case where one is able to establish a valid relationship between these observed behaviors and the considered knowledge itself. This relation between behaviors and knowing is crucial. It has been hidden as a result of the fight against behaviorism, but it has always been implicitly present in educational research at least at the methodological level. The key issue is that the meaning of a piece of knowledge cannot be reduced to behaviors, but on the other hand meaning cannot be characterised, diagnosed or taught without linking it to behaviors.

Such a link was clearly pointed out by Schoenfeld in 1987, in a book he edited under the title “*Cognitive Science and Mathematics Education*”. In the Introduction to the book he associates the Cognitive Science approach to an effort toward a more detailed description of problem-solving behaviors so that they could be then taught and reproduced. This position of Schoenfeld is synthesised by the following indication about his own research at this time: “My intention was to pose the question of problem-solving heuristics from a cognitive science perspective: What level of details is needed to describe problem-solving strategies so that students can actually use them?” (*ibid.* p.18). Such a *problématique* opens on two essential questions:

- On one hand, to which extent a finer granularity of a description would guarantee a better reliability of the transfer from one operator to an other? Or, rather, for any competency, does it exist a level of granularity, which gives an intrinsic guarantee for the efficiency of such a transfer?
- On the other hand, to which extent a finer description of problem-solving behaviors informs about the relationships between behaviors and knowings?

Concerning the second question, we must notice that Schoenfeld himself finally suggests that this relationship is essential in the chapter of his book devoted to constructivism—but may be without drawing all the consequences from this remark.

The question of the relationships between behaviors and knowings is considered as fundamental to the *theory of didactical situations* (Brousseau 1997)<sup>2</sup>. One of its postulates is that each problem-situation demands on the part of the student behaviors which are indications of knowing. This fundamental correspondence, established case by case, is justified by the interpretation of problem-situations in terms

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<sup>1</sup>We follow the choice made for the translation of Brousseau’s work (1997), to use *knowing* as a noun to distinguish the students personal constructs from *knowledge* which refers to intellectual constructs recognised by a social body. This intends to keep the distinction made in French between “*connaissance*” and “*savoir*”.

<sup>2</sup>For the convenience of the English speaking reader, we will take all the references to Brousseau's contributions to mathematics education from the book published in 1997 by Kluwer, but it should be noticed that the work we refer to was primarily published between 1970 and 1990.

of game, and by the interpretation of behaviors in terms of indication of strategies the adapted nature of which must be demonstrated in the model or representation attributed to the student (Brousseau 1997, p.215). This postulate is shared by some approach in Cognitive Science: “All behavior implies a knowing”, writes Pichot (1995, p.206)<sup>3</sup>. Indeed, this postulate justifies most of our experimental research since students’ behaviors are the source of the corpus on which we perform our analysis. But to “cut out” a behavior from the observation of a so-called reality, which could be a classroom or a laboratory experiment, is both a methodological and a theoretical problem as Robert emphasizes (1992, p.54). An observed *behavior* is not given by the “reality” but taken out of it as *a result of a decision taken by an observer*.

If a behavior is the description of material relationships between a person and her environment, then it depends on the characteristics of the person as well as on the characteristics of her environment. A good example is the case of instruments which at the same time facilitate action if the user holds the required knowing, and on the other hand limit this action because of their own limitations (Rabardel 1995, Resnick & Collins 1994, p.7). Actually, one should notice that these limitations could be related to material constraints as well as to the knowings involved in the design of these instruments.

“Person” and “environment” refer to complex realities of whose not all aspects are relevant for the type of questioning we are considering. What is of interest for us is the person from the point of view of her relationship to a piece of knowledge. For this reason we will refer from now on to the *subject* as a reduction of the person to her cognitive dimension. In the same way, we are not interested by the environment in all its complexity, but only by its features which are relevant with respect to a given piece of knowledge. We will call *milieu* such a subset of the environment of a subject; the milieu is a kind of projection of the environment onto its epistemic dimension.

Indeed, in the case of mathematics, knowings are not only the consequences of the interaction between a subject and a material milieu, but they involve also interactions with systems of signifiers produced by the subject herself, or by others. We must then extend the classical idea of *milieu* in order to integrate symbolic systems and social interaction as means for the production of knowings. This is the meaning of Brousseau’s proposal to *define the milieu as the subject’s antagonist system* in the learning process (Brousseau 1997 p.57). So, we do not consider a knowing as a property which can be ascribed only to the subject, nor only to the milieu<sup>4</sup>. On the contrary we consider it as a property of the interaction between the subject and the milieu—its antagonist system. This interaction is meaningful because it succeeds in fulfilling the necessary conditions for the viability of the *subject/milieu system*. By viability we mean that the subject/milieu system has a capacity to recover an equilibrium following some perturbations; what implies that the perturbation is recognized by the subject (for example, a contradiction or an uncertainty). In some cases the subject/milieu system may even evolve if the perturbations are such that this is necessary. This is, in other words, a formulation of Vergnaud’s postulate that problems are the source and the criteria of knowings (Vergnaud 1981 p.220).

*Problem* means here a more or less serious perturbation of the subject/milieu system. The existence of a knowing can then be evidenced by its manifestation as a problem-solving tool, what is reified as behaviors of the subject/milieu system as it overcomes the perturbations in order to satisfy its constraints of viability. These constraints do not address the way the equilibrium is recovered but the criterion of this equilibrium (we could also say that there is not only one way to know.) Following Stewart (1994 pp. 25-26) we would say that these constraints are *proscriptive*, what means that they express necessary conditions to ensure the system viability, and that they are not prescriptive since they do not tell in details in which way an equilibrium must be reconstructed (and we may add here that the descriptions searched for by Schoenfeld sound more prescriptive than proscriptive).

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<sup>3</sup> A statement which echoes the classical Piagetian position that Furth (1969) reminds us in his presentation of the theoretical foundation of constructivism: Piaget “identifies active intelligence with behavior” (focusing on the structural characteristic of adaptive behaviors, *ibid.* p.170). What means that “intelligence is not a thing that causes intelligent behavior; it is intelligent behavior in its active structure and essential aspects, which we for the sake of verbal economy call intelligence” (*ibid.* p.175).

<sup>4</sup> Taking into account a remark from one of the referees on a previous version of this article, it appears necessary to acknowledge clearly the close relationship between what is stated in the article and Piaget constructivism. But a key difference must be emphasized: our focus is on the whole system [S<>M] and not on one of its parts. In other words, our concern is not to know “how the subject think”, but to be able to give account of the whole system in a way relevant to a didactical project—the cognitive system we consider is not S but [S<>M].

We can now propose a definition of the meaning of a knowing which can be pragmatically and efficiently used in a didactical *problématique*.

*A knowing is characterised as the state of dynamical equilibrium of an action/feedback loop between a subject and a milieu under proscriptive constraints of viability.*

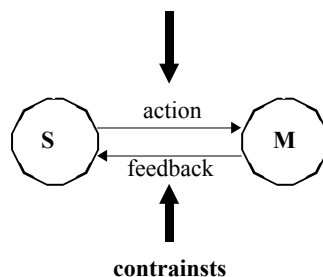


Figure 1.

Following the didactical *problématique* we are interested by the nature of the proscriptive constraints that the subject/milieu system must satisfy. Among these constraints, not known exhaustively, we can mention two that are specific to didactical systems: time constraints and epistemological constraints (Arsac *et al.* 1992). The former are due to the way schooling is organized (duration of school life, organization of the school year, organization of the lessons, etc.). The latter is due to the existence of a knowledge of reference which underlies any content to be taught and which *de facto* provides criteria to the acceptability of any learning outcome.

The role of the teacher, with respect to a given content to be taught, is to organize the encounter between a subject and a milieu so that a knowing—which can be seen as acceptable with respect to the didactical intention—can emerge from their interaction. Such an encounter is not a trivial event. To be in an environment is not enough for the student to be able to “read” in it the milieu relevant to the teaching purpose (Balacheff 1998). To select the relevant features of the environment, to identify the feedback and to understand it with respect to the intended target of action is not self-evident. The means for the teacher, to succeed in this task is to construct a situation that allows the devolution to the students of both the milieu and the relevant relationships (action/feedback) to this milieu. But the didactical intention of such a situation can act as a constraint; this is the case when the student believes in a teacher expectation, what could modify the nature of the subject/milieu system equilibrium and then the nature of the related knowing. But this is the basic complexity of didactical systems.

Learning is a process of reconstruction of an equilibrium of the subject/milieu system which has been lost following perturbations of the milieu, or perturbations of the constraints on the system, or even perturbations of the subject itself (modification of his or her intentions, or as a consequence of a brain disease, etc.). The didactical *problématique* considers the case of perturbations provoked on purpose, with the intention to stimulate learning. The indicator of a perturbation is the gap, recognized by the subject, between the expected result of an action and the actual feedback from the milieu. This means on one hand that the subject is able to recognize the existence of a gap not acceptable with respect to her intention, and on the other hand that the milieu can provide identifiable feedback.

Sometimes the subject does not identify a gap whereas we, as observers, identify that one should have been recognized. We call this unnoticed gap an *error* when it is the symptom of a knowing, that is: the symptom of the resilience of a previous equilibrium of the subject/milieu system.

As it is now well accepted, not all errors are negative. For the construction of certain knowings it is even better to be aware of some errors and to overcome them. For example, uniform convergence is a very powerful concept whose construction often proceeds via errors due to a common idea of conservation of certain properties of functions (like the statement—false in general—that the limit of a series of continuous function is a continuous function). But if research has been able to show in some cases that erroneous knowings could be necessary to the meaningful learning of some concepts, it has left open the question of the control on the means allowing their overcoming.

The knowing source of errors could be locally refuted, but it could keep some validity even with respect to the knowledge of reference, which may be seen as providing to it a domain of validity (the

concept of uniform convergence allows to express the domain of validity of the so-called principle of conservation of some properties). But even when this erroneous knowing is rejected and replaced by a new one, it may keep a pragmatic validity (decimal numbers are not natural numbers with a dot, but to consider them as such is quite useful insofar as computation is concerned). The consequence is the possibility of earliest knowings persisting despite their refutation, simultaneously with new knowings—what is likely to instigate yet the idea of an incoherent and unstable knowing subject.

## 2. AN EPISTEMOLOGICAL PROBLEM

### 2.1. COHERENCE AND SPHERE OF PRACTICE

“In the diagram of the calendar, the complete series of the temporal oppositions which are deployed successively by different agents in different situations, and which can never be practically mobilized together because of the necessities of practice never require such a synoptic apprehension but rather discourage it through their urgent demands, are juxtaposed in the simultaneity of a single space. The calendar thus creates *ex nihilo* a whole host of relations [...] between reference-points at different levels, which never being brought face to face in practice, are practically compatible even if they are logically contradictory.” (Bourdieu 1990, p. 83).

This explanation of the paradox of the co-existence of a rational thinking and of knowings, which looks contradictory from the observer’s point of view, can be extended to the case of a single agent observed in different situations. The core of this explanation is time on one hand, and on the other hand the diversity of situations. Time organizes the subject’s actions sequentially in such a way that the contradictory knowings are equally operational because they appear at different periods of his history—contradictory knowings can then ignore each other. The issue of the diversity of situations introduces an element of a different type. It is a possible explanation insofar as one recognizes that a knowing is not of a general nature but that, on the contrary, it is related to a specific and concrete domain of validity on which it is acknowledged as an efficient tool. This emphasizes that transfer from one situation to another one is not an obvious process, even if in the eyes of an observer these situations are isomorphic. Following Bourdieu, we will refer to *sphere of practice* in order to designate these domains of validity mutually exclusive in the history of the subject. We may say that in a sphere of practice the rational subject is reconciled with the knowing subject.

We must insist on the fact that the contradictions, which are evidenced in this way, are recognized as such by an observer who is able to relate situations that are seen as independent and completely different by the subject herself. But, nevertheless, in the *observer referential system*, these states of the observed subject/milieu system should be labeled in the same way. So, one may like to speak of the subject knowing of decimal numbers, of continuity of functions or of line reflection even if later on one would complain that this knowing is not coherent.

Indeed, to accept the existence of contradictory knowings seems to refute the theoretical principle of mental constructs as products of a process of adaptation ruled by criteria of reliability and of adequacy to problem-solving or task-performing. This raises problems in mathematics education, for which solutions have been looked for in different directions. We will here consider the most significant one and its evolution.

### 2.2 FROM MISCONCEPTIONS TO CONCEPTIONS AS KNOWING

In a survey she presented at the 1986 annual meeting of the *American Educational Research Association*, Confrey links the development of research on misconceptions to the acknowledgement of a failure of teaching: despite all efforts, many students held major misconceptions in mathematics and science. She notices that mathematics education community had a rather pragmatic approach to this problem:

“misconceptions were defined empirically as documented failures of large numbers of students to solve problems which appeared to be related to fundamental concepts.” (Confrey 1986 manuscript p.4).

In a paper published later (Confrey 1990), she distinguishes different approaches of this question<sup>5</sup>. In all of them the child-student is seen as a subject fundamentally different from the adult-expert who appears

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<sup>5</sup> They are the Piagetian genetic epistemology, the scientific epistemology and the information processing approach. Each of these three approaches aims at providing a *problématique* for the cases when the students’ conceptions appear to be in contradiction with shared and recognized knowledge (Confrey 1990 p.4).

as the owner of the knowledge of reference<sup>6</sup>. But this view does not exclude the recognition of some sort of cognitive legitimacy of these misconceptions<sup>7</sup>:

“[...] a child may not be ‘seeing’ the same set of events as a teacher, researcher or expert. It suggests that many times a child’s response is labeled erroneous too quickly and that if one were to imagine how the child was making sense of the situation, then one would find the errors to be reasoned and supportable” (ibid. p.29)

Indeed, within the student frame of reference—as opposed to an external one—conceptions fall under the common rules of knowing:

“a misconception does not require the postulation of an inadequate ‘picture’ of the world; it does require the notion of a successful completion of a number of problems wherein the cycle of problem formulation (expectation), problem-solving (action) and problem reconstruction (re-viewing) are successfully carried out.”

In other words: *a misconception has a domain of validity*, otherwise it would not exist as such. So, there is a very short distance between a misconception and a knowing. The key difference is that for a misconception there exists a refutation which is known at least to the observer. But even when ascribing to a misconception the status of a knowing, what leads most authors to abandon the word itself, it remains as a corollary of the initial definition the existence of an *intrinsically correct knowledge of reference*—although such a position is clearly refuted by our current knowledge of the history of science and mathematics. Let us notice, at this point, that considering that students’ knowing and knowledge of reference are of a different nature, has as a consequence the impossibility of a model, which would be a tool to give account of both.

Bachelard (1938, p.13) wrote in a nice manner that reality is never what one could believe, but is always what one should have thought of<sup>8</sup>. This statement, stated in the first half of the century, expressed that knowing is always in progress. If we accept this, errors witness the inertia of the instrumental power of a knowing which has proved itself by its efficiency in enough situations, and its organism likelihood in an environment which changes it and which changes itself in its turn. One may notice that constructivism did not immediately recognize this nature of error, or at least less than one might think. For example, Aebli (1963), who developed the application of the psychology of Jean Piaget to didactics, characterized errors as witnesses of a student’s misunderstanding or habits<sup>9</sup>. Thus the erroneous reactions, which they could provoke in some problem-solving situation must be studied in details with students so that they understand the reasons why some processes are not correct, and so that they capture differences and relations between the correct reaction and the error (ibid. p.101). To a certain extent we may suggest that in such a view the student is a cognitive subject but not yet a fully knowing subject:

“>From the functional level, the child is identical to the adult, but with a mental structure which varies depending on the stages of development” (Piaget 1969, p.224). Engaged in a construction process the child is always obliged to accommodate herself to an external reality, to the peculiarities of the environment from which she has to learn everything (ibid. p.225).

The content of the child mental structure has not yet completely the status of a knowing, even though all theoretical ingredients exist to allow to consider it as such. The Copernican revolution was not achieved at the beginning of the 70's.

The main evolution of the 80's, was to recognize that errors are not only the effect of ignorance, of uncertainty, of chance, but the effect of a previous knowing which was interesting and successful, but which now is revealed as false or simply not adapted (Brousseau 1997, p.82). In France, one of the first works in this paradigm (Salin 1976) proposed cognitive characteristics of errors essential to the development of the theory of didactical situations: on one hand an error is a point of view of a knowing about another knowing (possibly for a subject, the evaluation of an ancient knowing from the point of

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<sup>6</sup> In the case of the former one speaks of “naive theory”, “private concepts”, “beliefs” or even of the “mathematics of the child” (Confrey 1990 p.29)

<sup>7</sup> This remark, made in the case of the scientific epistemology, is in fact valid for all the three mentioned approaches, even for the Piagetian approach as we emphasise it in the following paragraph.

<sup>8</sup> “Le réel n’est jamais “ ce qu’on pourrait croire ” mais il est toujours ce qu’on aurait dû penser” (Bachelard 1938, p.13).

<sup>9</sup> “réflexes conditionnés déclenchés par un signal inducteur” (Aebli 1963 p.53).

view of a new one) , on the other hand an error can be identified only if the feedback from the milieu can be “read” by the subject as the indication of a failure (a non satisfied expectation).<sup>10</sup>

The main difference between the previous position and the current one lies in their epistemological meaning; the status of knowing is different in each case. The first position implies the existence of a knowing-of-reference general and true (a knowledge, as it were). The second position requires only to establish a relationship between two knowings with the idea of an evolution, without judgment on them. Any knowing is what it is, whether it appears to be erroneous or not, partial or ill adapted; it is first of all the result of an optimal adaptation of the subject/milieu system following criteria of adequacy and of efficiency.

As a consequence of the nature of the subject/milieu system, any knowing has a provisional character, or rather, any knowing could be revisited, its domain of validity can be modified as a result of some perturbations which it would be otiose to claim that they are unlikely. Indeed, one recognizes here the strong relation, which links “knowing” and “problems” for which this knowing is a tool (allowing to get back to an equilibrium).

### 3. CONCEPTION, A FORMAL CHARACTERISATION<sup>11</sup>

The word “conception” has been used for years in research on teaching and learning mathematics, but it has been used as a common sense notion rather than explicitly defined. It functions as a tool but its definition remains implicit; it is not taken as a object of study as such (Artigue 1991, p.266). According to Artigue’s analysis, “conception” refers to a local object; in this sense its epistemological status does not really differ from the one of the word “misconception”. There is a need for a better-grounded definition of conceptions, and for tools allowing us to analyze their differences and commonalties. A need already noticed by Vinner (1983, 1987). This section aims at proposing a solution to this problem in the form of a formalization of the notion of “conception”. Then, in the last sections of this paper, we will analyze the current literature about students’ conceptions of function as a case for a first illustration of the usefulness of our proposition.

This formalization should be a way to overcome the contradiction pointed out above, namely that the multiplicity of a subject’s knowings can prove contradictory with respect to some given reference. So, we will also propose definitions of the terms “knowing” and “concept” as abstract entities whose differences lie in their functions and relations.

We call conception C a quadruplet  $(P, R, L, \Sigma)$  in which:

- P is a set of problems;
- R is a set of operators;
- L is a representation system;
- $\Sigma$  is a control structure.

The informed reader will recognize, underlying the three first components, the key features identified by Vergnaud (1991 p.145)<sup>12</sup> in order to characterize a concept; we have introduced the fourth one for reasons we explain hereafter. The very first question of any researcher in mathematics education will be that of knowing how to relate this formal definition with the “reality” he or she is faced to. We will consider this point for each of the four elements of the definition.

The question of the concrete characterization of *the set P of problems*, is complex. One option would be to consider all the problems for which the considered conception provides efficient tools to elaborate a solution. This is the option suggested by Vergnaud in the case of additive structures (1991 p.145). This

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<sup>10</sup> The thesis of Brousseau at the beginning of the seventies, goes beyond the fact of recognizing that the mental constructs source of errors are knowings. It states that some of these knowings likely to be falsified are necessary to learning: the trajectory of the student may have to pass by the (provisional) construction of erroneous knowings because the awareness of the reasons why this knowing is erroneous is necessary to the construction and understanding of the new knowing. Following Bachelard (1938 pp. 13-22), Brousseau calls *epistemological obstacles* these compulsory gateways to new understanding: “A piece of knowledge, like an obstacle, is always the fruit of an interaction between the student and her surroundings and more precisely between the student and a situation which makes this knowing ‘of interest’. In particular, it stays ‘optimal’ in a certain domain defined by the numerical ‘informational’ characteristics of the situation.” (Brousseau 1997, p.85).

<sup>11</sup> Although rather complete, the presentation is here limited to the main outlines of the formal characterization of conceptions. The reader may be interested to visit the related web site [www.conception.imag.fr](http://www.conception.imag.fr).

<sup>12</sup> This definition proposed by Vergnaud was in fact coined at the beginning of the 80's.

solution is out of reach in most cases. Another option could consist of considering a finite set of problems with the idea that other problems will derive from them. This is the solution proposed by Brousseau (1997, p.30). If Vergnaud's option fails to help, Brousseau's option leaves open the question of establishing that a generative set of problems can be constructed for any conception. Instead, we propose to adopt a *pragmatic* position, deriving the description of P, in an empirical way, from the characterization of situations allowing to diagnose students' conceptions. This approach can be strengthened by the analysis of historical and actual uses of mathematics (e.g. Sierpiska 1989, Thurston 1994, d'Ambrosio 1993, Lave 1988, Nuñez *et al.* 1983).

The question of the concrete characterization of *the set R of operators* is more classical. Operators are means to obtain an evolution of the relations between the subject and the milieu; they are the tools for action. Operators could be "concrete", allowing to perform actions on a material milieu, or "abstract", allowing to transform linguistic, or symbolic, or graphical representations. So, an operator could take the form of functionality at the interface of a software or of a syntactic rule to transform an algebraic expression, or it could even take the form of a theorem in an inference.

*The representation system L* consists of a repertory of structured set of signifiers, of a linguistic nature or not, used at the interface between the subject and the milieu, supporting action and feedback, operations and decisions<sup>13</sup>. Just to mention few examples: algebraic language, geometrical drawing, natural language, but also interfaces of mathematical software and calculators are all examples of representation systems. Whatever it is, depending on the state of the subject/milieu system, the representation system must be adequate to give account of the problems and to allow performing operators.

The last dimension of a conception, *the control structure*  $\Sigma$ , is constituted by all the means needed in order to make choices, to take decisions, as well as to express judgment. This dimension is often left implicit although one may realize that the criteria which allow to decide whether an action is relevant or not, or that a problem is solved, is a crucial element of the understanding of a mathematical concept. We would suggest that in the Vergnaud proposition the control structure is implied by his reference to theorems-in-action or to inference (Vergnaud 1991 pp. 141-142), which are meaningful notions only to the extent that they are associated with the recognition that the subject has procedures to check that her actions are legitimate and correct. After Polya and a long tradition of research on meta-cognition, Schoenfeld (1985 pp. 97-143) has shown the crucial role of control in problem solving. More recently, Robert (1993) emphasized the role of meta-knowledge demonstrating *the need to treat control structures as such*. Indeed this is directly related to a *problématique* of validation, which is intrinsically related to understanding (Balacheff 1987 p.160).

It is important to insist on the fact that this characterization of a conception is not more related to the subject than to the milieu with which he or she interacts. On the contrary, it allows a characterization of the subject/milieu system: the representation system allows the formulation and the use of the operators by the active sender (the subject) as well as the reactive receiver (the milieu). The control structure allows to express the means of the subject to decide of the adequacy and validity of an action, as well as the criteria of the milieu for selecting a feedback.

## 4. THE CASE OF FUNCTIONS

### 4.1 A THEME EXTENSIVELY STUDIED

Instead of giving many examples, which we could explore only superficially in the limited space of the present article, we have chosen to investigate in a manner as precise as possible just one case, namely the case of "functions".

The theme of function has been extensively studied. It exists a lot of bibliographical references, all very different from each other. On the other hand, the notion of function is at the intersection of several mathematical areas (numbers, limit, algebra, etc.) and requires considering several representation systems

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<sup>13</sup> Following the remarks from one of our referee we must emphasize that we are aware of the difficulty which could be raised with the use of the word "representation", especially when it is read in the light of a psychological *problématique*. We do recognize that any "symbol as representation needs a living person who constructs the representation, or in comprehending reconstructs it" (Furth 1969 p.93). But we are here concentrating on the system formed by a knowing subject and a milieu, and not exclusively on one of them. Indeed, representation in this sense—that is in a semiotic sense—is the basic support of the observed behaviors. We do not mean to reduce either the subject or the knowledge to the signifiers.



(graphical representations, symbolic language, etc.). Such a rich context will help to demonstrate the benefit one may expect from the modeling of conceptions we propose.

It is classical nowadays to consider *a priori* the following categories of conceptions of function:

- Function as a correspondence "law" (a function expresses the correspondence between two sets, an element of the first set being associated with a unique element of the second set)
- Function as symbolic expressions (a formula, an algebraic expression...)
- Function as a graphical object

The two first formulations come from Vinner and Dreyfus (1989 pp. 359-360). As the reader may know, these authors consider other concept images of "function", such as "relation of dependence", "rule" and "operation". Other authors introduce other categories like "ratio and proportion" or "functional dependency" (Sierpiska 1989) or conception as processes (Breidenbach *et al.* 1992). These categories can be seen as refinements of the more general ones mentioned above. Because of the fragility of the means we have to ascribe a conception to a student, we chose here to remain with the three main definitions.

The methods usually used to ascribe a conception to a student consists more often than not in analyzing interviews or questionnaires which asked students whether it exists a function corresponding to a given specification (e.g. Vinner and Dreyfus 1989 p.359) or modeling some situations (e.g. Breidenbach *et al.* 1992 p.279), or even addressing directly the question "what is a function for you?" (Vinner and Dreyfus 1989 p.359). What students produce then is rather difficult to analyze. For example, one may wonder how it is possible to distinguish precisely between the category "dependency relationship between two variables" ("relation of dependence") and the category "something which relates the value of x to that of y" ("rule"). The issue pointed here is on one hand that of the way data are collected and its effect on the diagnostic of conceptions, and on the other hand, that of the way in which these conceptions can be described. We will come back to this question later.

The categories we have selected can be seen as invariant in the mathematics education literature, they in fact correspond to the three main representation systems associated to "function"—whether one considers research in mathematics education or research in history of mathematics. Indeed, it is by the historical analysis, using the classical works of several historians (Kline 1972, Smith 1958, Kleiner 1989, Edwards 1979), that we will introduce a first proposition for modeling the conceptions of "function".

## 4.2 CONCEPTIONS OF FUNCTION FROM AN HISTORICAL PERSPECTIVE

A good starting point to identify the main conceptions of "function" in the course of the history of mathematics, is to distinguish them by means of the main system of representation they implemented.

One of the most ancient sign of the existence of function are tables and their uses. For example, Ptolemy (in the *Almagest*) knew that positions of planets change with time and compiled astronomical numerical tables (Youschkevitch 1976, p.40-42). In the 10<sup>th</sup> and 11<sup>th</sup>, precise tables were also used in astronomy by Arabs. Tables go with locating an isolated number by another number (or quantities), and so, the idea of variable is not yet present.

The association of a curve and a table plays a critical role in formulating and solving the problem of determination of the trajectories of the planets. Kline, for example, indicates that Kepler has improved the computation of the position of planets essentially by adjustment of geometrical curves and astronomical data, without any theoretical reference to explain why the trajectories are elliptical (op. cit. p.336). The validity of the conjectured trajectories was then essentially related to the precision of the measurement of the planet positions and to the choice of a familiar geometrical object, the ellipse that permitted to describe the universe with simple mathematical laws. Kline notes that most of the functions introduced in the XVII<sup>e</sup> century were first studied as curves (ibid. p.338). In fact, curves, as trajectories of moving points, were the main object of study for mathematicians of the XVII<sup>e</sup> century (Kleiner 1989, p.283).

The creation of the symbolism of algebra (Viète 1540-1603, and later Descartes, Newton, Leibniz) has been decisive for the development of functions. The separation of the study of function from geometry is credited to Euler. Kleiner (ibid. p.284) emphasizes that the "*Introductio in analysin infinitorum*" (1748) offers an entirely algebraic approach with not a single picture or drawing. The function was presented as the central object of the Analysis. The analytic characterization of functions received a strong formulation by Euler, who stated that a function is an analytical expression formed in any manner from a variable quantity and constants (ibid. p.404).

Although considering function as an analytical expression proved to be a powerful tool, it caused contradictions and was inadequate to solve some problems of the 18<sup>th</sup> (e.g. the controversy of the vibrating strings). Euler formulated in 1755 a general definition of a function expressing the notion of

dependence between variable quantities and the notion of causality (Dhombres 1988, p.45). Such a definition opened new potentialities and difficulties which stimulated many discussions up to the 20<sup>th</sup> century (Monna 1972).

Each of these conceptions<sup>14</sup> can be characterized by a quadruplet as we have defined it above:

- *Table* conception:  $C_T=(P_T, R_T, Table, \Sigma_T)$ ,
- *Curve* conception:  $C_C=(P_C, R_C, Curve, \Sigma_C)$ ,
- *Analytic* conception:  $C_A=(P_A, R_A, Algebra, \Sigma_A)$ .
- *Relation* conception:  $C_R=(P_R, R_R, L_R, \Sigma_R)$

The cases of *Table*, *Curve* and *Algebra* conceptions are sufficient to illustrate our purpose, and so will not develop in details the case of the *Relation* conception. The words "Table", "Curve" "Algebra" are used to refer to the corresponding representation systems (what is not the case for the *Relation* conception) that are characterized by their specific syntactic rules and their own criteria for validity. The question is then to examine for  $C_T$ ,  $C_C$  and  $C_A$  whether the sets  $P$ ,  $R$  and  $\Sigma$  show significant differences depending on the conception they contribute to describe. We will consider this question from the point of view of the control structures.

The conception  $C_T$  has essentially empirical foundations. Indeed, the validity of a table depends on the precision of measurement and of the related computations against the requirements of a given circumstances. In the case of the conception  $C_C$  the validity must be evaluated against the quality of the interpolations and predictions that the ellipse allowed. Therefore the corresponding control structure  $\Sigma_T$  is fundamentally of an empirical nature, as well as for  $\Sigma_C$ . They must provide tools that allow verifying the precision of tables or of curves with reference to the observations and to the measurements, which have been carried out. The corresponding spheres of practice depend in an essential manner on the quality of rather concrete productions. Indeed, in the beginning of the XVIII<sup>o</sup> century an important problem was that of long distance navigation, coasts being out of sight<sup>15</sup>.

Thus the set of problems  $P_T$ , as well as  $P_C$ , were dominated by practical questions and  $R_T$ , as well as  $R_C$ , included—but not exclusively—techniques of measurement, of computation and of drawing. At this point we may suggest that the ellipse of Kepler's first law was a geometrical object, ideally conceived of but empirically used when constructing curves in order to have access to the object and for predictions. A curve was not the graphical representation of what we acknowledge nowadays as being the graph of a function considered as a relationship between entities (numbers or even quantities).

The analytic conception  $C_A$  is of a different nature. In our view it introduces a rupture in the epistemology of functions. A function defined by an analytical expression does not need to refer to an experimental field (either of natural phenomena, or of mechanical drawings). It can be studied for itself, as Euler expressed by presenting functions as the object of study of the Analysis (*Introductio in analysin infinitorum*). That does not mean that the *problématique* of modeling no longer plays any role, but rather that it is no longer central and does not characterize the conception. A purpose of the Analysis of the 18<sup>th</sup> (and of the 19<sup>th</sup> and 20<sup>th</sup>...) was the resolution of functional equations, which were of a great importance for Physics (Dhombres 1988), and the developments into infinite series, which played a central role as operators ( $R_A$ ) in these resolutions. The corresponding control structure  $\Sigma_A$  depends on the specific characteristics of Algebra as a representation system and on the operators it allows to implement. Computation of symbolic expressions and mathematical proof are the key tools to decide whether a statement is valid or not. Indeed, symbolic representations are not the only ones to be available and to be used. To any  $C_A$  function can be associated a graph, that is a set of pairs  $(x, y)$  in the Cartesian plane (where  $y$  is the value of the function for a given  $x$ ). This possibility suggests often close relationships between  $C_C$  and  $C_A$  that raises the question of the relationship between graph and curve. While the graph is a possible representation of a function, displaying phenomena that algebraic expressions do not easily demonstrate (for example the intersection of two lines), a curve is rather an evocation of the trajectory of a mobile point or of a geometrical object, as Kline expresses when describing Newton's conception (op. cit. p.339).

The general solution of partial differential equations expressing the vibrations of a finite string, because of the initial conditions, induced Euler to consider arbitrary functions which had not necessarily

<sup>14</sup>□ We do not claim that these three conceptions are sufficient in order to give account of the historical development of "function". They are keystones from which the historical analysis may be carried out and organised.

<sup>15</sup> □ In the sixteenth century the methods of [computing longitude] were so inaccurate that navigators were often in error as much as 500 miles" (Kline 1972, p.336). In 1712, the British government established a commission for the discovery of longitude

an analytic representation. The existence of such arbitrary functions was controlled by physical arguments and was related to the various possible initial forms of the string. The emergence of a  $C_R$  conception of function required then to develop new modes of representation  $L_R$  and new controls structure  $\Sigma_R$  in order to define what such functions could be and in order to work with them without any reference to an analytical representation. It took two centuries.

It is important here to emphasize that conceptions as we describe them, differ the one from the other in an essential way. In order to illustrate this, consider various definitions of the derivative as given by Thurston (1994):

- “Symbolic : the derivative of  $x^n$  is  $nx^{n-1}$ , the derivative of  $\sin(x)$  is  $\cos(x)$ , the derivative of  $f \circ g$  is  $f' \circ g \cdot g'$ , etc.”
- “Geometric : the derivative is the slope of a line tangent to the graph of the function, if the graph has a tangent.”
- “Approximation : the derivative of a function is the best linear approximation of the function near a point”.

Thurston proposes that each of the different definitions expresses “ways of understanding a particular piece of mathematics”, but that they are in fact equivalent provided that one makes the effort to formulate them with enough precision (Thurston 1994, pp.4-7). If this could be the case for an expert who actually sees a metaphoric relationship in between the different settings evoked, it is far from being the case for a learner or even from the point of view of the history of mathematics along which these definitions appeared. Conceptions such as we described them do not verify the property of equivalence suggested by Thurston.

Coming back to the case of function, the issue is not that of the precision in formulating a definition, nor is it a question of language. The conception of curves as trajectories of a point, ascribed to Newton by Kline (1972 p.339) is fundamentally different from the Dirichlet conception of a sub-set of the Cartesian product of two sets satisfying given constraints (which guarantees univalence). The crucial point is that “function” does not refer to the same object in the two cases but to objects that are different in essence.

Without going too far in the discussion of these points we must notice the fact that each representation system we consider, taken for itself (with its semiotic characteristics) has a different displaying power (which can be defined as the capacity to show what should be shown, or possibly to hide). These differences can be better understood by considering the operators, which can be implemented and the corresponding control systems.

Let us take again the case of tables ( $C_T$ ). A table is a rather particular way to represent a function insofar as it does not tell anything beyond the values that it displays. Although it is the first means of representation used, the crucible within which were shaped quite a number of functions—Kline (1972, p.338) reminds us that the table of sinus was known with a great precision long before the associated curve became a mathematical object. The validity of a statement based on the manipulation of a table depends, as we already emphasized, on the precision of the computation or of the precision of the observations that produces its elements. Here it appears that the control structure  $\Sigma_T$  heavily depends on the relation to a domain exterior to mathematics for which the considered conception gives the means for modeling. In the same manner,  $R_T$  includes both mathematical operators and operators specific to collecting data.

### 4.3 STUDENTS' CONCEPTIONS OF FUNCTION

We will focus on students from the secondary and post secondary levels. These students constitute the bulk of the population considered in the literature. They all have some knowledge of Algebra and they all have been exposed to classical elementary functions.

The study of Vinner on students' concept image of function is classical. Vinner identified eight components of students' conceptions of function of which we reproduce here the main features:

- “The correspondence which constitutes the function should be systematic, should be established by a rule and the rule itself should have its own regularities”;
- “A function must be an algebraic term”;
- “A function is identified with one of its graphical or symbolic representations”;
- “A function should be given by one rule”;
- “Function can have different rules of correspondence for disjoint domains provided that these domains are regular domains (like half lines or intervals);
- “A rule of correspondence which is not an algebraic rule is a function only if the mathematical community officially announced it as a function”;

- “The graph of a function should be regular and systematic”;
- “A function is a one-to-one correspondence”.

These components of the concept image of function (which we took from Vinner 1992 p. 200) result from the investigations carried out by the author with students in the Jerusalem area at the beginning of the eighties. Since this seminal work of Vinner in 1983, *these features have been confirmed as being largely common to students all over the world* (Tall 1996, pp.297-301).

As they are presented to the reader, these features are not yet organized into conceptions. In particular one misses indications about their domain of validity as well as about the way they could be implemented in a problem-solving situations. One can notice that several of them are tightly related to one representation system, algebraic or graphic. For this reason, we consider representations as a privileged entrance point for the search for a characterization of the different conceptions.

From the students’ point of view, the idea that a graph-curve should exist in relation to an algebraic expression is central, both representation having to conform certain constraints<sup>16</sup>. By the expression ‘graph-curve’ we refer to two different entities that must be distinguished. This is well expressed by Sierpinska when she introduced the distinction between synthetic views and analytic views of curves<sup>17</sup>:

“Curve analytical view: a function is an ‘abstract’ curve in a system of coordinates; this means that it is conceived of points  $(x, y)$ , where  $x$  and  $y$  are related to each other somehow.” (ibid. 1989 pp.18).

“Curve, a synthetic view: [...] function is identified with its representation in the plane; it is a curve viewed in a concrete, synthetic way.” (Sierpinska 1989 p.17).

This distinction reminds us of the one that is usually made in mathematics between curve and graph: curve referring to a geometrical object, and graph referring to a representation of a function in the graphical representation system (one plots a graph). But the distinction made by Sierpinska here may not be exactly this one since she added then that the “relationship (*between  $x$  and  $y$ , the analytical view*) can be given by an equation. But the curve does not *represent* the relation. Rather, it is represented by the equation.” (ibid.). This remark indeed draws our attention to the confusion likely to be made by students between the geometrical setting and the calculus setting (in Douady's sense, Douady 1985), which is facilitated by the common graphical representation system used by both of them.

Thus, we suggest considering two types of student conceptions: the Curve-Algebraic and the Algebraic-Graph. We will consider them in a more precise way and describe them using the tools just presented:

- *Curve-Algebraic* conception:  $C_{GA}=(P_{GA}, R_{GA}, \text{Graphic-Algebraic}, \Sigma_{GA})$ ,
- *Algebraic-Graph* conception:  $C_{AG}=(P_{AG}, R_{AG}, \text{Algebraic-Graph}, \Sigma_{AG})$

These two conceptions share the same representation system, both algebraic and graphic, but these systems have different degrees of importance in both of them (this is suggested by their different order in their given name). In the case of the Curve-Algebraic conception, the criterion is that the curve must be represented by an equation as Sierpinska wrote—but we would prefer to say that it should have an algebraic representation rather than an equation. In the case of the Algebraic-Graph conception the criterion is that the algebraic representation must be associated with a graph which one must be able to *plot*. The distinction between graph and curve is not very easy to make because they implement the same representation system. The reasons for this difficulty are the same reasons why students have difficulties in recognizing in which setting they are working (and so, in knowing what is legitimate or not). It is by looking at the rules-tools that both conceptions require, and looking at their control structure that we can shape the distinction.

We would suggest that the task is rather difficult when one can just rely on the evidences provided by tasks where students are directly asked to answer the question “what is a function?”, or where they are invited to decide if graphs or descriptions of correspondences represent functions (ex : “Does there exist a function all of whose values are equal to each other?” (Vinner and Dreyfus 1989)). One can notice two important points about such experiments:

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<sup>16</sup> There is a rich literature on understanding functions and their representations. A common distinction is made between the process/object aspects of functions (Dubinsky and Harel 1992, Sfard 1991). Those aspects can be related to the students' uses of different representations of functions (Schwingendorf, Hawks and Beineke 1992, DeMarois and Tall 1999). The distinction between the global approach and the point-wise approach to functions (Bell and Janvier 1981, Even 1998) is critical as well, showing in particular that the capacity to deal with (the graphical and the algebraic) representations of functions or to move from one representation to another is related to the flexibility in using different approaches to functions. Because of the restricted place allowed in this paper, all those aspects cannot be developed and we limit ourselves to the specific case of the graph/curve distinction.

<sup>17</sup> Although she keeps using the word “curve” in both cases

- Firstly, they show the distance between the answers to the question “what is a function?” and those to the tasks requiring a decision on descriptions of correspondences. This distance evidences the relationship between conceptions and problems. Different tasks (like providing a definition of or describing a function) may call for different conceptions. This means that students’ conceptions are less accessible in statements about a concept, than in problem solving situations involving the concept. For this reason, the characterization of conceptions requires evidencing the relations between conceptions and problems.
- The second point concerns the type of tasks proposed to students. In all of them, students do not need to perform actions in order to produce a solution but rather to activate control operators in order to produce a decision—like the ones mentioned by Vinner and quoted above. Indeed, Vinner emphasizes the fragility of this type of investigation. He observed a large number of what he calls irrelevant reasoning and that he defines as follows: “[...] justification given by the student because he or she assumed it was the right thing to say (and no meaningful thought was involved) will not be considered as a relevant reasoning.” (Vinner 1992, p.206).

This can be analyzed under the light of the very clear description by Castella (1995) of the situation she used to explore students’ conceptions of tangent. Castella writes that she assumes that the drawings she proposed to students are “straightforward”, that their approximate character (they are sketches of functions) does not hide any surprising feature—the functions represented are what they seem to be<sup>18</sup> (ibid. p.21). One can then legitimately think that the observed situations may well depend heavily on the quality of the *experimental contract*. The researcher claims that the investigation targets the students’ conceptions, but it may be the case that what is observed is students’ *contingent opinions* and not students’ *conceptions* as expected. Students’ answers might be a way to please the teacher/observer expectation—this is exactly what Vinner feared. But indeed, how could it be different? Especially in the graphical representation system, tasks are provided to students using what we suggest to call *function icons* instead of functions’ graphical representations. In order to understand this, we invite the reader to consider such tasks from the point of view of the nature of the feedback the students can expect from the environment provided by the situation in which they are involved. They cannot perform any relevant action on the graphical representation since these representations are just sketches (see Castella pictures or Vinner and Dreyfus tasks evoked above).

We come here to a point where what should be discussed is the nature of the problems for which students’ conceptions provide tools allowing to propose reasonable solutions (at least in the student’s eyes). Indeed, *the characterization of a conception should not be separated from the characterization of the problem situation which allowed to evidence it.*

To go beyond the definition and investigate more problem oriented situations is indeed what Vinner (1992) intended when he asked students to decide on the continuity or on the differentiability of a function (ibid p.202-210). He concluded from his study that for students (i) for a “function to be continuous is the same as being defined and to be discontinuous is the same as being undefined at a certain point” (ibid. p.205), or (ii) “continuity or discontinuity is related to the graph” [e.g. ‘a function is continuous because its graph can be drawn in one stroke’] (ibid. p. 206), or (iii) “there is a certain reference to the concept of limit” [e.g. ‘The function is continuous because it tends to a limit for every x’] (ibid. p.207). Whereas the direct question “what is a function?” gave essentially an indication of possible elements of the control structures, we get here an insight about the tools students may have available.

Some of these tools clearly relate to a Curve-Algebraic conception. For example, drawing the graph  $[y \equiv 1 \text{ if } x > 0 \text{ and } y = -1 \text{ if } x < 0]$  is a tool which allows to control the aspect of the graphical object (the graph shows a disruption) and so allows to conclude that the function  $f(x) = |x|/x$  is discontinuous (Vinner’s task B1). Others tools relate to an Algebraic-Graph conception (like the ones involving a criterion of limit).

Vinner did the same with the derivative, identifying (i) correct algebraic characterizations, or (ii) descriptions of symbolic manipulations to be performed, or (iii) correct definitions within the graphical representation system. But the author seems to lack effective tools for describing the so-called incorrect solutions. This results in the use of attributes like “vague”, “fuzzy” or “meaningless”. Altogether these vague and fuzzy students answers represent 46% of the sample of 119 students. We must realize here that students were confronted with graphical representations of functions and asked *to decide* and *to*

<sup>18</sup> “Les fonctions représentées sont bien ce qu’elles ont l’air d’être”, writes Castella.

*justify* whether they were continuous or not. In the case of derivative they were directly asked *to define* it, answering the question “what is a derivative?” Again we observe the fragility of the data gathered, insofar as these tasks may well not reveal students’ conceptions of derivative but their possible *opinion* offered in a rather embarrassing situation.

In order to decide or to justify a statement, one has to mobilize one’s conception in a different way, more operational, than to define it, so to say, *in abstracto*. The point here is that what we learn from a student discourse or production must be analyzed against the characteristics of the situation in which this discourse or production is produced. We may even go farther claiming that *conceptions* and *problems* are dual entities (Balacheff 1995). This means that in order to characterize students’ conceptions one should provide them with meaningful problem situations, with enough complexity so that they can involve their conception in a significant way, demonstrating for us the tools they use and the nature of the control they involve in the task<sup>19</sup>.

Artigue (1992 p.130) gives us an excellent example of the benefit of coming back to the students’ situation characteristics when she analyses, in the case of the qualitative approach to differential equations, the *false theorem*: “if  $f(x)$  has a finite limit when  $x$  tends toward infinity, its derivative  $f'(x)$  tends to 0”. She wrote:

“In the field of differential equations, monotonicity conceptions may especially act as obstacles as on one hand, effective predictions are implicitly based on the extra-hypothesis of monotonicity and on the other hand, theorems have to get free of these extra-hypothesis. Let us be more explicit:

When sketching solution curves, we draw the simplest one compatible with the identified set of constraints, but in doing this, we add extra constraints concerning the convexity that can be expressed roughly in the following way: convexity has to be the least changing possible or, in algebraic terms, the sign of  $f''$ , for a solution  $f$ , has to be the most constant possible. So,  $f'$  is implicitly the most monotonic possible.” (ibid. p.130)

She concludes that the mentioned false theorem “can be seen as an instantiation of such an extra-hypothesis: if for  $x$  large enough,  $f'$  is monotonic,  $f'$  has necessarily a limit (finite or not) and the unique limit compatible with an horizontal asymptote is 0. In other words, by adding the condition  $f'$  monotonic, this false theorem becomes a true one” (ibid.) Such an analysis of a tool (which is the actual status of theorems in students’ practice) used by students, evidences the interaction of the graphic and the algebraic representation system and the role played by the characteristics of the sphere of practice (a role which is in fact recognized by Vinner when he points the phenomenon of “compartmentalization”).

Let us then come back to the Curve-Algebraic conceptions and the Algebraic-Graph conceptions. These conceptions will be distinguished not by the representation system (either graphic or symbolic) they mobilize, but by the type of tools and controls they involve in problem-situations. This means that to characterize these conceptions we need to be able to describe the set of operators  $R$  and the control structure  $\Sigma$  in relation to the set of problems  $P$ .

To state what is  $P$  in both cases remains an open problem for research in mathematics education even in this domain so much investigated. One may observe, at this point, that history is not of a great help. Actually students’ conceptions are very difficult to analyze against what history teaches us about the evolution of the concept of function. And indeed we would be very cautious with the idea that the “historical study of the notion of function together with its epistemological analysis helped us to analyze the student’ mathematical behavior” (Sierpinska 1989 p.2). It is clear that the epistemological analysis is an essential tool, but the historical analysis may induce a view of the notion of function which hides the role played by the modern school context. The historical analysis will delineate the notion from the mathematical point of view, from the cognitive point of view we must be prepared to see things in a rather different way. Actually Sierpinska acknowledged that “the students’ conceptions are not faithful images of the corresponding historical conception” (ibid. p.19). For example, one of the questions one has to consider is that of knowing what could be the essential difference between the students algebraic conceptions and the “corresponding” historical conceptions. It is also striking that tables play a very limited role if at all in the situations involving functions: if they are present it is in relation to concrete situations in which the aim is less to analyze a function than to analyze the data (the function is seen as a tool for data analysis).

The spheres of practice of students are radically different from the ones of the mathematicians that historians consider. The didactical system has first introduced students to “good” functions, mainly

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<sup>19</sup> Note that we do not pretend that this is enough to solve our diagnostic problem, but that it is a necessary condition.

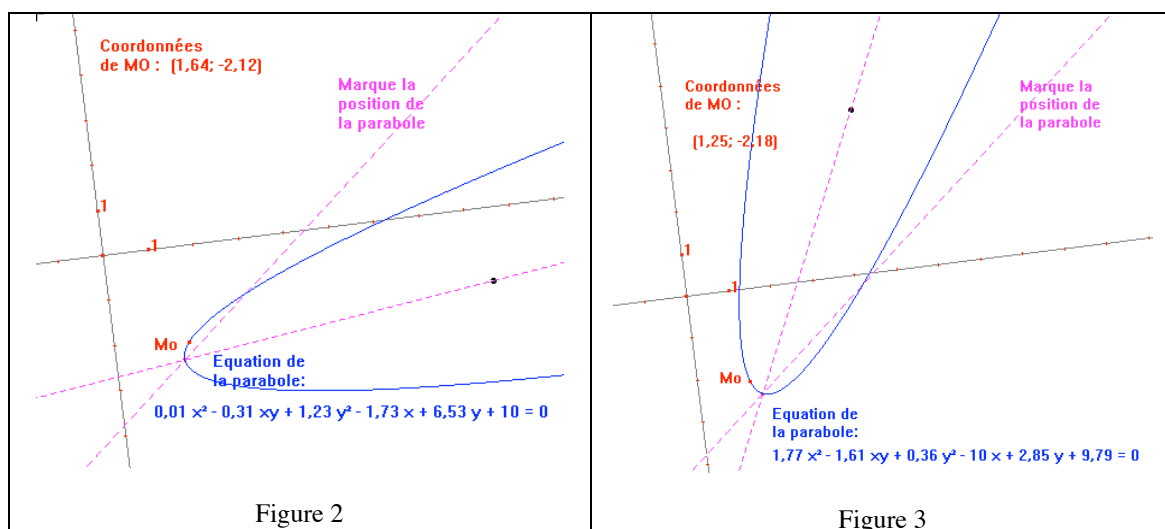
playing with two different settings: algebraic and graphical. As Sierpinska (1989 p.17) noticed, shapes of graphs of elementary functions can function as prototypes of conceptions. Depending on the curriculum they have been exposed to, students have available more or less sophisticated tools in order to analyze some elementary algebraic formulas, and to describe the behavior of the corresponding functions. These spheres of practice may be described in detail following a close analysis of textbooks which are available to students (see for a first investigation in this direction, Mesa 2001).

#### 4.4 EXAMPLE OF A STUDY: THE CURVE-ALGEBRAIC AND ALGEBRAIC-GRAPH CONCEPTIONS

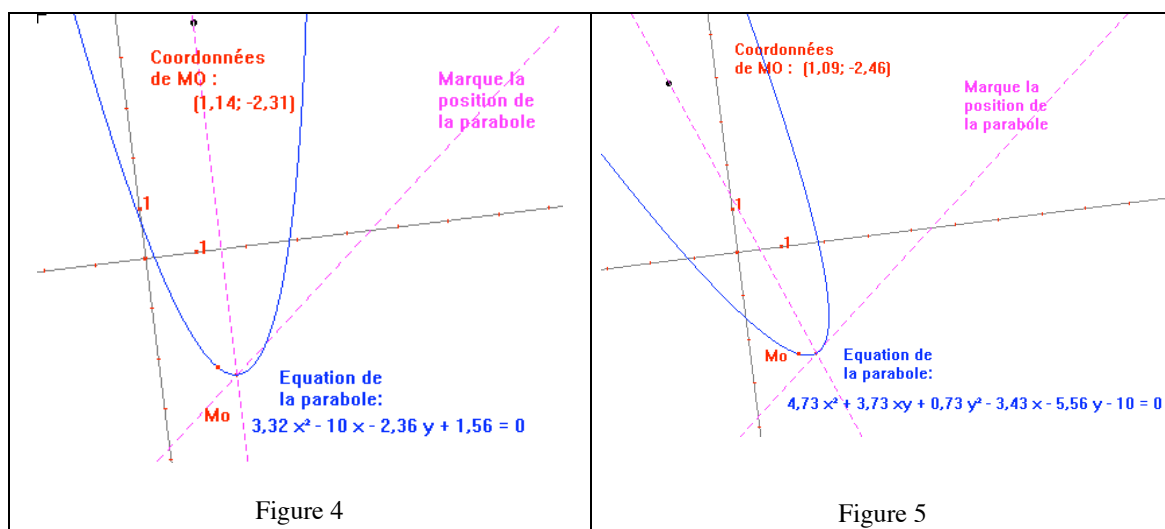
We will illustrate the use of the proposed framework for the characterization of students' conceptions of functions, taking the case of students from the 12<sup>th</sup> grade (last year of the secondary school in the French educational system). We wanted to go farther in the understanding of the relations between the graphical and the algebraic representation system. Our previous study gave rise to the characterization of two conceptions of functions: the *Curve-Algebraic* and the *Algebraic-Graph*. We would like to demonstrate in this section how the algebraic and the graphical representation system, and so, the rules-tools and the controls required in the problem, lead to the differentiation of these conceptions.

We used the dynamic geometry environment *Cabri-Geometry II*<sup>20</sup> (hereafter named *Cabri*) in which one can construct curves and display the dynamical relation between their graphical and their algebraic representations (Gaudin 2002).

A parabola in a system of coordinates was presented to the students. This parabola could turn around its vertex and could be manipulated, the associated equation being updated dynamically during these manipulations (see figures 2 to 5).



<sup>20</sup> *Cabri-Geometry II* is a dynamic geometry environment distributed by Texas Instruments.

Different positions of the parabola in *Cabri*

We asked students to give the equation of the tangent to the parabola at point  $M_0$  and to draw this tangent (the students knew how to draw with *Cabri* a straight line of which the Cartesian equation is known).

*Situation 1:* the parabola could be manipulated by moving its axis (grasping the black point, see Figures 2-5) so that it could reach a “vertical” position (Figure 4)—in this position the parabola is the graph of a quadratic function.

*Situation 2:* the movements were constrained, the parabola could not turn completely around the vertex, hence it could not reach the vertical position as in situation 1; and so students could not get a “familiar” quadratic function.

12<sup>th</sup> grade students were supposed to know:

- that a parabola which equation is  $y = ax^2 + bx + c$  is the graph of the function  $f$  defined by  $f(x) = ax^2 + bx + c$ .
- that an equation of the tangent to the graph of a function  $f$  at point  $M_0(x_0; y_0)$  is  $y = f'(x_0)(x - x_0) + f(x_0)$ ,  $f'$  being the derivative of  $f$ .
- some rules of differentiation.
- nothing about conics.

In order to frame the analysis of students problem-solving activity we develop, we sketch hereafter the strategies which could be expected in both situations:

Situation 1:

- A control on the position of the parabola allows reaching the vertical position to get the graph of a quadratic function  $f$ . Its algebraic representation is  $f(x) = (1/2.36)(3.32x^2 - 10x + 1.56)$  (see Figure 4).
- The property “An equation of the tangent of a graph of a function  $f$  at point  $M_0(x_0; y_0)$  is  $y = f'(x_0)(x - x_0) + f(x_0)$ ,  $f'$  is the derivative of  $f$ ” is an operator which processes the algebraic expression of  $f$ .
- One gets the equation of the tangent of the parabola ( $y = -1.01x - 1.17$ ) and draws the tangent with *Cabri*.

Situation 2:

- The parabola can not reach the vertical position anymore, but, in the horizontal, position it is the graph of a quadratic function in another system of coordinates defined by the same origin, the point  $(0; -1)$  as the unit on the  $x$  axis and the point  $(1; 0)$  as the unit on the  $y$  axis (see Figure 6).
- The new system is associated to the change of variable  $X = -y$  and  $Y = x$ . This change is a tool to know the equation  $3.08X^2 - 2.04Y - 9.23X + 10 = 0$  of the parabola in the new



system and to get the algebraic representation  $f(X) = (1/2.04)(3.08X^2 - 9.23X + 10)$  of the function of which the parabola is a graph in the new system.

- Like in situation 1, the property “An equation of the tangent of a graph of a function  $f$  at point  $M_0(x_0; y_0)$  is  $y = f'(x_0)(x - x_0) + f(x_0)$ ,  $f'$  is the derivative of  $f$ ” is an operator which processes the algebraic expression of  $f$ . One gets the equation of the tangent of the parabola ( $Y = -1.20X + 3.06$ ). The change of variable  $X = -y$  and  $Y = x$  is a tool to know the equation of the tangent in the initial system of coordinates:  $1.20y - x + 3.06 = 0$ . One draws the tangent with *Cabri*.

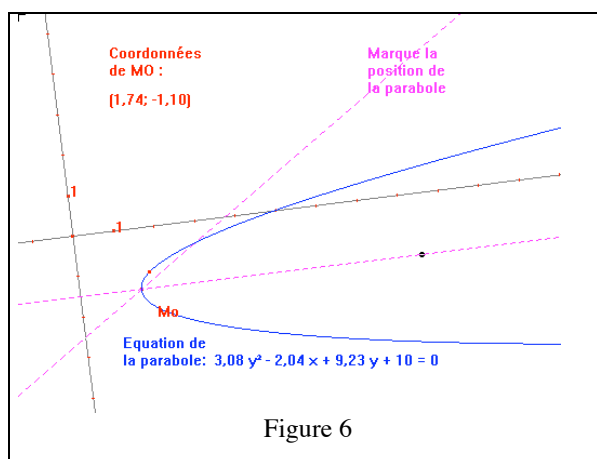


Figure 6

The drawing of a parabola on the screen can be conceptualized as a *graph* or as a *curve*. As reminded in section 4.3, the *graph* of a function  $f$  is the set of points  $(x, f(x))$  in a system of coordinates; a *curve* is a geometrical object which can be drawn independently from the existence of a system of coordinates. Considering the parabola as the graph of a quadratic function, is essential in order to associate controls to the tools involved in the above mentioned strategies.

We will present some aspects of the analysis of the observed behaviors of two students André and Rémi who worked together, as well as of Loïc and Sylvain. We have chosen these two case studies because they illustrate in a very good way both conceptions: *Curve-Algebraic* in the case of André/Rémi, and *Algebraic-Graph*<sup>21</sup> in the case of Loïc/Sylvain. From these students' point of view, getting the equation of the tangent of the parabola meant “deriving the equation”<sup>22</sup> and using the operator “if  $f$  is a function of  $x$  then the equation of the tangent at the point  $M(x_0, f(x_0))$  is  $y = f'(x_0)(x - x_0) + f(x_0)$ ”. As long as the parabola was not the representation of a graph, students could not perform the rules of differentiation they knew (and so, use the operator and get the equation of the tangent). Their decisions and actions to make the equation in accordance with these rules revealed different controls and tools involved in the problem.

Identifying the control used by André/Rémi or Loïc/Sylvain to establish if an equation was derivable or not in situation 1, provides means to discriminate both conceptions.

For André/Rémi, the equation could be derived if its form is  $[y = \langle \text{an expression of } x \rangle]$ . Since it was not the case, they tried to change the position of the parabola. They moved it to the vertical position. They chose this position while controlling the equation which, as a result, became simpler and simpler ( $y$  disappears in the equation: *that's good*<sup>23</sup>, the coefficient of  $xy$  decreases: *that's perfect...*). They obtained an equation that they called an “*arranged equation*” that could be *derived*.

Sylvain/Loïc found the equation on the screen really complicated (*monstrous*). They controlled that *the equation was not the representation of a function* because the graph did not pass the vertical line test (*there were two values for one  $x$* ). They stated that the equation could not be

<sup>21</sup> For more details, see Nathalie Gaudin (1999).

<sup>22</sup> We will specify what means “derive the equation” for each students' pair.

<sup>23</sup> Students' exchanges are written in italic.

derived. The control on the derivability was based on the form of the equation which had to be  $[y=(\text{some } x, \text{ no } y)]$ . Sylvain anticipated that  $xy$  and  $y^2$  should disappear in the equation when the parabola would reach the vertical position, so they decided to get this position.

The actions of both student pairs in situation 1 are the same: moving the parabola to the vertical position and getting the equation of the tangent as described in the expected strategy. But the controls and the associated representation system differ in an important way. André/Rémi's controls refer essentially to the algebraic representation system: getting an adequate symbolic writing of the equation. Sylvain/Loïc's controls refer to the graphic and algebraic representation system: satisfying the vertical line test to get an equation of the form:  $[y=(\text{some } x, \text{ no } y)]$ .

These controls play an essential role in students' decisions in situation 2. Noticing that the parabola cannot reach any more the vertical position, both students' pairs had to change their strategies.

André/Rémi decided to move the parabola to the horizontal position because the equation appeared then to be the simplest one they could expect. The obtained equation  $(3.08y^2 - 2.04x + 9.23y + 10 = 0$ , see Figure 5) was described as "nice", as "the best". But this equation was not yet in accordance with the rule of differentiation and students proposed to *change x with y and y with x*. Again, the control is essentially algebraic. They doubted the legitimacy of such a change, and then, they proposed to change the system of coordinates into the new one according to the mapping of variables:  $X = -y$  and  $Y = x$  (which is almost the one suggested by *changing x with y and y with x*). They got an expression that they could derive  $(f(x) = (1/2.04)(3.08X^2 - 9.23X + 10)$ —one can notice that they wrote "x" and not "X", in " $f(x)$ ". The important point here is the meaning students ascribed to the change of system. They did not relate this changing to the other objects of the situation: they used the initial coordinate of  $M_0$  in the operator "if f is a function of x then the equation of the tangent at the point  $M(x_0, f(x_0))$  is  $y = f'(x_0)(x-x_0) + f(x_0)$ ", the equation of the tangent was not rewritten in the initial system prior to drawing the tangent (as described by the "expected" strategies), thus the tangent was invalidated. Clearly, the change of system was not associated to a control in the graphical representation system. It was only associated to a control on the symbolic writing of the equation conforming to  $[y \square (\text{expression of } x)]$ .

Loïc moved the parabola to the horizontal because this position appeared to him to be *better than any position*. Sylvain did not consider this position as a better one to solve the problem. This opinion was consistent with the vertical line test he used in situation 1 and is confirmed by the presence of  $y^2$  in the equation (*I don't see... how will you cop with the  $y^2$ ?*). He proposed to change the system of coordinates: the new proposed system being the one in which the parabola satisfied the vertical line test, and so, the one in which the parabola was the graph of a function. Thus, unlike the other student pair, it was a control in the graphical representation system which led to choose the new system of coordinate. This control allowed a distinction graph/curve more effective than "getting a nice equation". "More effective" means that this control was associated to new tools used by the students: reading the coordinates of  $M_0$  in the new system, getting the equation of the tangent in which the parabola is a graph, and then writing this equation in the initial system to draw the tangent.

The identification of different conceptions in these two case-studies is possible by looking at controls, at tools and at the associated representation systems the students used in order to solve the problems. In the present situation, the actions we observed refer to rather different controls and do not define the same settings of work.

In the case of the *Curve-Algebraic* conception, the control associated to the operator "if f is a function of x then the equation of the tangent at the point  $M(x_0, f(x_0))$  is  $y = f'(x_0)(x-x_0) + f(x_0)$ " is an algebraic one. The parabola is a geometric object designated by an equation and some of its positions are more or less operational from the point of view of this control. Thus, the algebraic tools and transformations (change of variable) do not apply on the objects of the geometric setting (the parabola,  $M_0$ , the tangent straight line). Calculus is reduced to symbolic transformations.

In the case of the *Algebraic-Graph* conception, graphic controls (recognizing the parabola as a graph, reading the situation in a new system) are related to algebraic tools (the operator "if f is a

function of  $x$  then the equation of the tangent at the point  $M(x_0, f(x_0))$  is  $y = f'(x_0)(x - x_0) + f(x_0)$ , the change of variables that works on every objects of the system). Because of this relation, Calculus can be based on variables linked by an equation and/or a graph.

However, we have to be careful with the relation between conceptions and situations. It is important to check the characterized conceptions against classrooms practices. We have to define in which way the characterized conceptions work on the exercises proposed in classes and to which extent they are closely specific to the present situations. In other words there is still to confirm that both conceptions are associated to real spheres of practice.

## 5. CONCLUSION: CONCEPTION, KNOWING AND CONCEPT

There is not enough room within the limits of the present paper to give a detailed account of all the benefit we expect from the definition of "conception" we propose. Just as an example, will show how *the model allows expressing more precisely that two conceptions have the same content of reference*, or refer to the same object. To express such a relation between conceptions is necessary in order to allow the passage from the level of conceptions to the level of knowings. To do so, one needs a third party, an observer:

Let call  $C_\mu$  the conception of an observer (in other words, we have to make explicit a conception of reference), and  $C$  and  $C'$  two conceptions. If it exists a translating function  $f: L \rightarrow L_\mu$ , and  $f': L' \rightarrow L_\mu$  such that for any problem  $p$  from  $P$  it exists a problem  $p'$  from  $P'$  so that  $f(p)=f'(p')$ —and conversely—then we say that  $C$  and  $C'$  refer to the same content of reference from the point of view of  $C_\mu$ .

Note that to say that two conceptions have the same content of reference from the point of view of an observer does not say much about the nature of these conceptions; they could be wrong or right, efficient or not, more or less general... All kind of properties that we can express in a precise way within the model<sup>24</sup>, and which we may use to "compute" optimal learning situations.

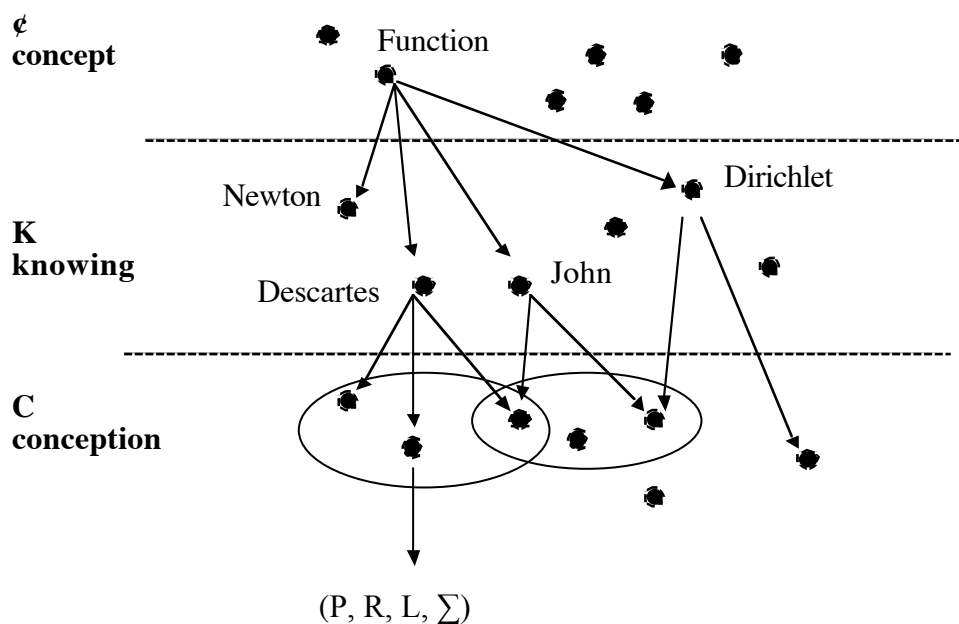
As a conclusion we will only indicate the progress we have made considering the fundamental problem described in section 2.

If we call *knowing* a set of conceptions which refer to the same content of reference, we can then speak of the domain of validity of a knowing (i.e. the union of the domain of validity of the related conceptions) but at the same time we can acknowledge the contradictory character of a knowing (one conception is false from the point of view of another one, both being constitutive of the *same* meaning -- cf. Balacheff 1995, p.233).

Moreover, we suggest to call *concept* the set of *all* the knowings sharing the same content of reference (Balacheff 1995, pp. 234-235). In the case of function, for example, investigating its meaning consists first in constructing a schema like the following and understanding the relationships at the level of the conception (is it possible, for example, to provoke the passage from one conception to another one?).

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<sup>24</sup> See Balacheff 1995, 1995a.



To some extent we are close to answer positively the question Biehler posed<sup>25</sup>: “Couldn't we say that the meaning of a mathematical concept is the synthesis of all its uses? ”. On the other hand, we are at some distance from the proposal of Sfard (1991) who wrote that “the word ‘concept’ (sometimes replaced by ‘notion’) will be mentioned whenever a mathematical idea is concerned in its ‘official’ form—as a theoretical construct within ‘the formal universe of ideal knowledge’; the whole cluster of internal representations and associations evoked by the concept—the concept's counterpart in the internal, subjective ‘universe of human knowing’—will be referred to as ‘conception’” (ibid. p.3). However, if the view of “concept” could be considered more pragmatic in our presentation, the view of “conception” we have developed is not so far away from the one Sfard holds.

To summarize, one can see that a conception is the instantiation of the knowing of a subject by a situation (it characterizes the *subject/milieu system in a situation*), or it could be considered as the instantiation of a concept by a the pair (subject/situation). From the relationships between conceptions induced by the definition adopted here, and from their properties, we can draw in a natural way properties and relationships between knowings, as well as between concepts.

We may have to emphasize that we have considered here the milieu, as well as the interactions between the subject and the milieu from the unique point of view of didactical situations—the teacher is not considered in this model. Indeed, the teacher must be considered as soon as we consider the conditions for a given student to come to an interaction with a milieu which we consider likely to allow some learning. This will be the core of further development.

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<sup>25</sup> We refer to a question posed by Rolf Biehler in a working paper he shared in the context of the BACOMET project IV “Meaning in mathematics education” (1996).

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