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Integrative theoretical framework

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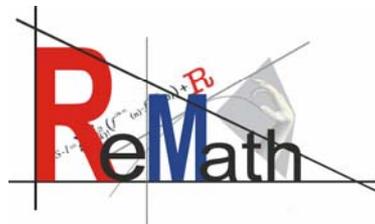
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Representing Mathematics with Digital Media



INTEGRATIVE THEORETICAL FRAMEWORK

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I. Introduction

At the origin of ReMath lies the fact that the huge efforts which have been performed in most European countries to improve mathematics education through the development and use of Interactive Learning Environments (ILE), have had a limited impact on the reality of school practices up to now. For more than 20 years, for instance, research literature has been evidencing the fact the new means provided by ILEs for representing and manipulating the abstract entities that the mathematical objects and processes are, offer new avenues for mathematical learning, but these evidences permeate educational systems poorly. For the ReMath partners, part of the difficulty comes from the fragmented character of the theoretical frames, which have been developed in order to approach learning and teaching processes in mathematics education. Part of the difficulty comes also from the insufficient attention that research tends to pay to contextual issues, both for design and use, and especially to the distance separating the experimental contexts where the learning potential of ILEs is generally evidenced from the context of ordinary classrooms.

Thus the ReMath project of looking for integrating perspectives in terms of theoretical frameworks, and the choice made to closely link their construction to the development of some specific ILEs and to experimentations of these carried out in realistic educational contexts. This is the task to be performed in the WP1 of ReMath, and this first deliverable gives account of the work carried out in that direction during the first six months of existence of this STREP.

The diversity of its theoretical frames can give the feeling that the field of mathematics education is something like a Babel Tower. This situation is more and more perceived as problematic for the development of research itself, and even more for the development of efficient links between research and practice. This is evidenced by the increasing number of papers, research projects and conferences addressing this issue, and trying to find ways to improve the current situation (see for instance the two recent issues of the *Zentralblatt für Didaktik der Mathematik* in 2005 and 2006¹, or the specific working group on these issues at the European Conference in Mathematics Education: CERME4 in 2005²). ReMath can be considered as one of such projects, with the specificity that, beyond the sole community of mathematics education, it also involves communities such as that of computer scientists or researchers in AIED and EIAH, and that its specific focus is on representations.

Looking for integrative perspectives raises some fundamental questions. What kind of integration can reasonably be aimed at? Does it make sense to look for a unified perspective, an overarching theory or meta-theory encompassing the different existing frames? Or is such a perspective unreasonable, due to the incommensurability of most of the existing theoretical frames? What can only make sense would be then to look for structures and languages in order to better understand the characteristics of the corresponding approaches, to organize the communication between these, and to benefit from their respective affordances. If so, can we build such structures and languages, and how can we make these operational? These are the

¹ <http://www.fiz-karlsruhe.de/fiz/publications/zdm/zdm061a.html>

² <http://cerme4.crm.es/Papers%20definitius/11/paperswg11.htm>

questions we are facing in the WP1 of ReMath and that we began to address in the first six months of the project.

As mentioned above, issues raised by the diversity of theoretical frames are given increasing interest in the mathematics education community. Within the TELMA European Research Team of the European Network of Excellence Kaleidoscope, most ReMath partners have been working on these in the last two years, and they have obtained some preliminary results. It was thus decided to start the reflection by reviewing the work already developed in TELMA, complementing this review by the analysis of different complementary sources. We first selected the following sources:

- Two specific issues of the *Zentralblatt für Didaktik der Mathematik* (vol. 37.6 in 2005 and vol. 38.1 in 2006) devoted to theoretical considerations
- The proceedings of the working group of the European Conference CERME 4 (2005) devoted to the role and comparison of theoretical frames in mathematics education (Dreyfus & al., to appear)
- A chapter by Cobb entitled “Putting Philosophy to Work: Coping with Multiple Theoretical Perspectives”, to be published in the second NCTM Handbook of Research in Mathematics Education in 2006
- Two issues of the *Journal of Mathematical Behaviour* synthesizing the reflection on representations carried out within the working group on representations of the international group Psychology of Mathematics Education, during its many years of existence
- A chapter presenting the results of a meta-study on research and innovation related to the integration of computer technologies in the Second International Handbook in Mathematics Education (Lagrange & al., 2003)
- The report of a study group at the National Centre for Scientific Research in France about theoretical and methodological basis of EIAH design (Tchounikine, 2004)

The analysis of these sources together with the reflective analysis of TELMA work were used in order to make clear what exact aims would be given to the development of an integrative perspective, and the means we could use in order to have this integration as operational as possible. Simultaneously, we began to work on the connection to be developed with the different WPs of ReMath, elaborating specific methodologies for this purpose.

In this deliverable, we give a synthetic view of the work carried out so far and of its first achievements, before presenting how we see its future development. The core of the deliverable is structured into four main parts. In the first part, we present the results of the reflective work developed on TELMA activities. In the second, we illustrate the work carried out on the complementary sources mentioned above and its outcomes through four different examples. In the third, we introduce the structure we propose as a basis for an integrative perspective. In the fourth, we describe how the connection with the other WPs has been organized, developing especially WP3 and WP4 that we consider the most critical in terms of necessary connection as regards the near future.

II. A reflective analysis of the TELMA work

TELMA is a European Research Team of the European Network of Excellence Kaleidoscope including five teams with a strong tradition in the field of technology enhanced mathematics

learning, also engaged in ReMath. Its aim is to promote the construction of a shared scientific vision between the different teams, to favour the development of common projects, the building of complementarities and common priorities. As a first step in that direction, a description of the different teams was collaboratively prepared, structured along the following dimensions: scientific goals, theoretical frameworks of reference, ILEs³ designed and/or used, research methodologies, specific projects⁴. Then the teams decided to focus on three different themes: theoretical frames, representations and contexts, analysing and comparing their different perspectives on these through the study of a collection of selected publications.

This comparative study proved useful for improving mutual understanding but it also appeared that it had evident limits, due to the methodology used: the exact role played by theories remains largely implicit in most published papers, and the data one would like access to in order to understand this role better are not generally the data that are provided. In order to overcome these limitations, it was thus decided to complement this analysis by a cross-experimentation where each team would have to experiment an ILE designed by another team, in order to provoke deeper interaction between the teams. In this part, we develop a synthetic and reflective look at the work carried out so far in TELMA, looking at what it can offer to ReMath. In doing so, we draw on the work produced within the TELMA project but also elaborate on this with further reflection and insights that have been developed during the first six months of ReMath. We successively consider the work carried out on representations, contexts and theoretical frames by referring to the corresponding published deliverables, and we enter into more details as regards the on-going work on cross-experimentations not yet published.

II.1 Representations

Although we may speak of mathematical objects, these are not concrete objects. In order to operate with them they must be represented. This representation may be through the use of a wide range of semiotic systems. These include 'natural' language and the conventional systems of numeric and algebraic notation, graphs and diagrams, but may also include idiosyncratic systems developed for didactic purposes or invented by students themselves. While these different semiotic systems may appear to be used to refer to similar mathematical objects (for example, an equation, a verbal description, a graph and a table of numerical values can all represent the 'same' function), they allow different kinds of manipulations and have different potentials for meaning making (Duval, 2000; O'Halloran, 2005).

In the context of Mathematical education there are two distinct meanings of the term "representation". On the one hand there are *external representations* of mathematical objects and processes that may be encountered or produced by different participants (students, teachers, researchers, designers...), and on the other hand the *internal representations* or conceptions that they have of mathematical objects or processes. The introduction of a tool brings issues of representation to the fore. New experiences related to the use of the tool provide different resources that may influence the formation of internal representations. Digital media can be seen as providing new avenues for introducing students to mathematics. In particular, current research has highlighted the importance of features such as:

³ The more general term of ICT tool is used in TELMA reports instead of the term ILE in order to qualify the kind of technology developed or used by TELMA partners. In this report, we will use the term ILE which, in our opinion, qualifies better the kind of technology referred to or the term DDA used in the presentation of ReMath.

⁴ These documents and other TELMA productions are available on the TELMA website whose address is: www.itd.cnr.it/telma

programming through symbolic and eventually mathematical notation; dynamic manipulation; availability of multiple representations; control of computational objects (see, e.g., Noss & Hoyles, 1996).

The reflective work carried out within the TELMA teams highlighted differences that do not allow adoption of a unified terminology in approaching cognitive issues and in particular do not allow them use of the term internal representation in a consistent way. On the other hand, the teams found it possible to use the notion of external representation in a compatible way. In particular, they found they could converge on a few aspects related to this notion that are usable for framing their common research work, including cross-experimentation, and the development and integration of DDAs.

In educational research, the representation of mathematical objects was originally addressed through considering and identifying the misconceptions and misunderstandings that representations could generate. The situation has progressively moved, due to the increasing influence of socio-cultural approaches especially sensitive to the semiotic dimension of mathematical activity. It has also developed thanks to the introduction of new technologies offering multi-semiotic environments within which learners can engage in mathematical thinking *through* new kinds of representations and new kinds of access to conventional ones. In this case, research is showing that, far from being a source of misunderstanding, representing mathematics can instead become a means of generating and expressing mathematical meaning, especially when situated in a social, collaborative context. The ways in which the users of technological tools make use of representations in making mathematical and other meanings is thus also a common interest for the TELMA teams.

It is important to recognise that a particular representation may have different meanings for the designer and for the user. In particular, while a tool designer may have a clear idea of the ways in which a chosen representation relates to a specific mathematical object, this does not guarantee that a particular user will even see it as being mathematical at all. The relationship between representation and object is thus dependent on the perspective of the interpreter.

One of the aims of the TELMA project was to make explicit the different perspectives of the various teams on the issue of representation. This issue arose in three ways in relation to the goals of the various research teams:

1. the design of the tools used and the ways in which these provide representations of mathematical objects and activities and of the social dimension of teaching and learning;
2. the study of the ways in which students make use of representations of mathematical concepts during activities involving use of technological tools and the relationship of these to their construction of mathematical meanings;
3. the development and use of analytic frameworks for studying technological tools and the ways they frame the activity of students and teachers of mathematics. Such frameworks aim to provide means of systematising the ways in which researchers and designers conceptualise the representation of mathematical objects and activities. They may then be used to analyse the potentialities of tools for mathematics teaching and learning, to inform the design of new tools, and to understand better the ways in which existing tools are likely to be used by teachers and students.

While there are both theoretical and practical differences in the ways the TELMA teams addressed these issues, they share a vision of the importance of the choices of representation systems and forms of interaction incorporated into technological tools in influencing their

potential impact in educational settings. We summarize below the main similarities and differences observed according to these dimensions.

a) Concerns about representation in the design of tools

Mathematical discourse is multi-semiotic in the sense that it makes use of a range of different semiotic systems or registers each of which allows different forms of manipulation and has different meaning potential. Recognition of this has meant that the choice of representations has been an important feature of tool design for the TELMA teams.

Even where teams have developed tools addressing similar mathematical domains, their projects' different educational aims have led to significantly different choice of representations. For example, DIDIREM, Did@tic and Pisa/Siena have all been involved in the design of CAS-like tools and have in all cases employed conventional numeric and algebraic notation but, because each had rather different educational aims, transformations and relationships between algebraic expressions were represented in different ways in order to facilitate different types of problem solving and mathematical thinking.

In some cases the choices of forms of representation has been explicitly informed by theoretical perspectives. Among the TELMA teams, the concept of *microworld* is a basic common element of the framework of reference in the design of tools, though it does not have a single clear definition. At the core of the relationship between the user and the knowledge domain are the objects of the interface that are available to the user. Papert termed these *transitional objects* "standing between the concrete/manipulable and the formal/abstract" (as cited by Noss & Hoyles, 1996). Such objects are the means of interaction between the user and the environment, thus, between the learner and the knowledge domain. A microworld is thus an environment where exploration is possible thanks to transitional objects, but where such exploration is constrained in ways designed to promote learning. It includes a set of computational objects that represent, or are considered to embody, ideas and objects belonging to the target domain, or are conceived as mediating tools for the construction of a solution process (Bottino & Chiappini, 2002). These objects can be manipulated in some way by the user in order either to explore the microworld (and hence, by analogy, the target domain) or to develop solution strategies for problems set or selected by the user or by their teacher. From this perspective, the focus for design of a tool is on the nature of the computational objects which form the central elements in a microworld and the relationships between them, the choice of which is critical.

The NKUA-ETL and ITD teams both draw explicitly on theories of microworlds to frame their tool design, though with rather different outcomes. The NKUA-ETL team puts emphasis on mathematical formalism as a means of representing mathematical ideas. They perceive their *E-Slate* artefacts to provide opportunities for learners to engage in meaningful formalism through microworld-style activities. In contrast, the ITD design of *ARI-LAB-2* emphasises the use of representations of concrete objects to support students in the solution of arithmetic problems. They make available a set of microworlds within which students can visually represent and manipulate problem situations in a variety of concrete contexts that are meaningful also from a mathematical point of view.

Artificial Intelligence theories, for example the Rewriting Rule Theory (Dershowitz & Jouannaud, 1989), developed in the domain of computer science, also lay high importance on the representation of mathematical objects as a critical component of tool design. Employed by the Did@tic team in the design of *APLUSIX*, these result in the production of theoretically informed forms of representation, in particular informing the construction of rules for transformation of algebraic expressions. While the development of such rules is based on analysis of human behaviour, they may not match it exactly.

Another common concern of the TELMA teams is with the social dimension of learning and mathematical activity. This manifests itself not only in a research focus on the use of tools by learners and teachers in a social context (discussed in the next section) but also in the incorporation of features in the design of the tools themselves such as the construction of specific roles for teachers or the representation of the doing of mathematics as a collaborative, social activity. Such incorporation is prompted by theoretical perspectives on learning but the work of TELMA to this point suggests that the specific design of such representations is not yet explicitly theoretically informed.

In addition to recognition of the importance of the choice of representations from the point of view of affecting the potential for mathematical meaning making, we have thus found a common interest in considering the theoretical basis of the design of the following aspects - one related to the design of the tool and the other related to the design of its use:

- the representation of didactic interaction in technological tools (for instance, through collaborative tools, virtual tutor, feedback, assessment, hints, ...);
- the ways didactic interaction is represented in Scenarios describing the modalities of use of a tool.

b) Concerns about the ways learners make use of the representations provided by tools

All the TELMA teams have engaged in one way or another in evaluation of the use of technological tools by students and/or teachers. As part of this evaluation, some have a specific focus on students' use of representations of mathematical concepts and the role this may play in their construction of mathematical meanings. The close interaction between the representations provided within the design of a tool and the mathematical meanings that may be constructed by students is a component of several theories employed by the TELMA teams to conceptualise knowledge and learning and has been a focus of research into the use of tools. For example, the global goal of the research of the Pisa/Siena team has been to investigate the specific semiotic mediations offered by microworlds and the way these foster the construction of mathematical meanings by students. A focus of the Did@tic team has been the use of technological tools to assist in modelling students' conceptions, using the cKc model (Balacheff & Gaudin, 2002) within which the representational system is one component. The NKUA-ETL team has employed the construct of *situated abstraction* in order to understand student meaning generation when students interact with a technological environment, studying the ways students use representations while exploring microworlds.

Introducing a new tool not only may change the personal relationship of an individual to mathematics, but also may change the mutual relationships between participants in respect to mathematics. Focusing on the representations provided by tools, the Social Semiotic theory used by the IOE team emphasises the functionality of semiotic systems in constructing both the experiential world and interpersonal relationships and personal identities. Socio-cultural approaches to learning employed by several of the TELMA teams also demand that attention should be paid to the social context within which learners interact with technological tools and other artefacts. There is thus a strong interaction between the study of representations and of context that underpins the present ReMath project.

c) Analytic frameworks used for describing and studying representations

As can be seen from the previous sections, representations of mathematical objects and of didactic interactions must be considered central to the design of ILEs, to designing scenarios for their use, and to studying the potential and actual mathematical meaning making during

their use by students and teachers. While this has been a concern within TELMA, the development of analytic tools and frameworks for their principled study is at an early stage. Some aspects have been addressed within individual teams. For example, as mentioned above, the Did@tic team's design of representations of algebraic transformations is informed by theories from the field of Artificial Intelligence; within the DIDIREM team, the instrumental approach as presented in (Guin, Ruthven, & Trouche, 2004) has been used to construct systematic description of the algebraic objects and modes of representation for these offered by spreadsheets, situated with respect to the standard algebraic objects and representation modes used in paper-pencil environment (Haspekian, 2005); a particular focus of the IOE team is on using Social Semiotics and Systemic-Functional Linguistics to characterise the ways in which tools represent the nature of mathematics and mathematical activity and the representation of teaching and learning.

In preparation for the cross-experimentations carried out within TELMA two sets of questions related to representation were prepared. These addressed the characterisation of, respectively, the experiential and the interpersonal functions of the tool, viewed as a means of communication. The first set of questions sought to distinguish the ways in which the tools represent mathematics and mathematical activity:

- What is the target domain?
- What are the objects and ideas of the target domain that are represented in the tool? (And are there aspects of the domain that are not represented?)
- How are these objects and ideas represented? What physical forms do they have? What relationships are there between them? What behaviours and functions are attributed to them?
- How are objects manipulated? Are they manipulated directly, e.g. by dragging, or symbolically?
- Which aspects of an object may be transformed by manipulation and which are invariant?
- What is the “distance” between the objects and the means of manipulating provided by the tool and those used in paper-and-pencil based work within the target domain? (And what effects might such differences have on the meanings users may construct?)
- What types of problems in school/college mathematics lend themselves best to solution using the tools provided?

The second set addressed the ways in which the tool represents the didactic interaction:

- Who may pose problems to be solved?
- What choices are available to the user about strategies for solving problems and about the tools to employ in developing their solution strategy? Are these choices equally available to teacher- and student-users?
- What forms of feedback are provided?
- How are solutions validated and by whom?
- Is communication with other users (students or teachers) incorporated and represented within the system?
- If communication is incorporated, is it between teacher and student or between students? What kinds of things may be communicated and what constraints are there?

In practice, within the small-scale experimentation enabled by TELMA it was only possible to begin to address a small fraction of these questions (see below, part II.4). Moreover, the teams do not yet have a common set of analytic tools that would allow systematic comparison of analyses.

These questions were developed for the purpose of analysing existing technological tools but form a basis for adaptation to produce a framework for considering the design of tools and of scenarios for their use, which can be of interest for ReMath.

II.2 Contexts

The notion of context is meant to take account of the complex system of both immediate and broad goals, social and cultural values, individual and institutional relationships, tools and of those situational, social and cultural elements within which individuals act and which influence the individuals' activity itself.

There are two ways in which contexts are important in our work within the ReMath project.

- At a basic level there is a practical problem of how our group of researchers, working in different contexts, are able to understand and make use of each other's work. This addresses the problem of how a research study can be understood from outside the context in which it took place.
- At a theoretical level there is the question of how different theoretical frameworks address the nature of context and its effects on the different elements of the research system (design, use and research).

Both levels ultimately attempt to address the issue of how it might be possible to find a tool or a method to re-contextualise the use of an ILE by drawing from the experience in one context in order to use it in another. Within the TELMA project, the former level has been directly approached. The starting point is to remark that the notion of Learning Contexts as usually referred to in math education (i.e. "a means to describe and understand the learning environments where the use of technology takes place") is definitely too narrow. Teaching and learning involve complex processes, and bringing in technology adds even more complexity. In order to deal with such complexity, TELMA tried first to encompass the main aspects of the context which usually remain implicit and nevertheless influence

1. the ways the educational processes are studied;
2. the ways educational environments based on the use of technologies are generated and sustained and
3. the methods, collaboration and interaction and organizational structure behind teams' productions (research projects, DDAs, educational materials).

In more detail, within the TELMA project the work on contexts has been structured as follows:

- a) the setting up of clusters of contextual issues – formulated as questions - referring to research approaches, educational environments and material productions;
- b) the selection of questions for each cluster that seem relevant and shared by the teams;
- c) the description of the activities and approaches of each team according to this structure;
- d) the production of a synthetic description of the teams' work according to the same structure.

This resulted in a somewhat pragmatic definition of learning contexts and in the development of an analytical tool for identifying and describing contextual issues across the TELMA teams. Moreover the utilisation of this tool offers a deeper insight for analysing how contextual issues affect what knowledge, learning environment and products are constructed.

A number of contextual concerns were thus identified and structured in four different clusters:

- a) an educational environment cluster (addressing influence number 1);
- b) a socio-systemic cluster, which includes concerns about the relations and collaborations between the organizations and between the actors involved in the educational process (addressing influence number 2);
- c) a teacher education and support cluster (also addressing influence number 2);
- d) a technology design and development cluster, whose importance clearly emerges within the project where many teams are involved in design and development of technological tools (addressing influence number 3);

In educational research the socio-systemic and design aspects of contexts in particular are rarely explicitly addressed, in spite of their great impact on both the types of technology emerging and the kinds of use of such technology in the educational context (diSessa, 2004).

a) The context of educational environments

As argued at the beginning of this section, addressing the issue of context means addressing many questions influencing the research design and implementation, and usually left implicit. With respect to the contextual issues described so far, those related to the context of educational environments appear more directly linked to the theoretical frameworks which the research teams refer to. Indeed any theoretical framework even though not explicitly addressing them may provide elements to deal with issues like the social dimension of learning, or the role of teacher mediation, and so on.

In a sense this kind of cluster represents the point on which (usually) explicit elements of the theoretical frameworks and (often) implicit elements of context might hinge.

Within the TELMA project the following five sub-clusters were identified:

- social aspects of learning,
- nature of tasks,
- process of mathematical reasoning,
- teacher mediation,
- use of language.

We describe below how they were dealt with.

Social aspects of learning

The socio-cultural perspective in mathematics education looks at meaning making as a process of interaction between people and participation in communities and cultures.

The main question concerning the social dimension of learning is:

- What are the ways by which the social aspects are addressed (i.e. interaction, participation in communities, groups) in the classroom?

When the use of ICT tools is introduced in classroom activities, new elements have to be taken into account: the interaction between pupils, teacher and the tools themselves. As a consequence the complexity of the social dimension increases.

From the synthesis it emerges how the different ways of addressing the social nature of learning are filtered by different theoretical perspectives. For instance the DIDIREM team, focusing on using technology as a means to bridge the gaps between technique-theory articulations and student's practices, clearly refers to the anthropological approach whereas Siena team develops a Vygotskian approach centred on semiotic mediation. At the same time, consistently with the activity theory, ITD has been involved in designing a communication environment for joint problem solving activity as well as didactic practices supporting socially situated interaction and investigation.

Nature of tasks

An important shared aspect concerns the nature of the activities and tasks proposed to students.

- What is the nature and the type of the activities and tasks (structured, game-like, scenarios, projects) given to students?
- What characteristics of the activities support the generation of meanings?

The TELMA team activities range from well structured and strictly defined tasks aiming to identify students reasoning on specific curriculum based concepts to loosely defined exploratory activities aiming to elicit the generation of meaning in a constructionist or experimental or even playful way.

The Did@TIC-MeTAH team chooses the classic structured tasks given to students in their traditional curriculum and word problems. This team puts emphasis on supporting solution processes through the use of the Aplusix system providing several kinds of feedback. ITD exploits the action potentialities of microworlds and those of the Communication environment (one of the tools integrated in ARI-LAB), to design two different types of tasks: in the first type individual students solve traditional text-based problems through the use of microworlds while in the second type pupils are engaged in the construction of a problem solution exploiting the communication possibilities offered by ARI-LAB. The NKUA group has objectified the development of educational activity plans which develop from one document that is progressively modified into more discrete versions.

As before, the different approaches to the nature of tasks may be related to the specific theoretical frameworks of reference of each team.

Process of mathematical reasoning

The process of mathematical reasoning is recognized as central by all the TELMA teams, in one way or another. Within the project, the question is addressed to what extent the interaction of mathematical reasoning with various aspects of the context of learning situations is made explicit:

- How is the mathematical reasoning integrated in educational environments?

The TELMA teams addressed this issue according to different perspectives. The DIDIREM research studies, carried out about CAS and spreadsheets, show a positive influence of such professional environments on mathematical reasoning processes provided that a specific attention is given to instrumental genesis. According to the theory of didactic situations, the Did@TIC-MeTAH team sees Aplusix as a milieu for learning in which mathematical reasoning appears on the basis of contradictions, difficulties, and disturbing situations that

appear mainly as a result of the different categories of feedback. Drawing on the theory of embodied cognition, the ITD team has studied the role of the cognitive mechanisms underpinning interaction with the ARI-LAB microworlds in the production of meaning. From a socio-constructivist position NKUA keeps an open mind to pupil's ideas during their experimentation rather than taking the structure and concepts of a given curriculum as the starting point for student understandings.

Teacher mediation

The TELMA teams have recognized the importance of the teacher's role in technology based mathematical environments. The acknowledged need to address this issue explicitly has led to posing the following crucial question:

- How is the teacher's role related to different aspects of educational environments?

The TELMA teams deal with this issue in different ways according to different perspectives.

The Did@TIC-MeTAH has been focused on designing Aplux to provide facilities for teachers to build their own exercises and modules allowing them to observe the students' behaviour. Taking a similar perspective, the ITD team has designed tools to facilitate the teacher in the planning and management of the class activity. NKUA team has carried out research on teachers' practices in the classroom during infusion of pedagogical intervention as well as in teachers' engagement in experiential mathematics activities with exploratory software within innovative professional development courses. The Siena team specifically addresses the role of teacher within the frame of the Vygotskian notion of semiotic mediation.

Use of language

The TELMA teams have different ways of perceiving the role of the language in mathematical learning and refer to different theoretical frameworks to interpret the learning processes within contexts involving mathematical discourse. Thus the main question differently addressed by the teams:

- What is the role of language in educational environments?

The IOE group draw from the field of systemic functional linguistics (Halliday, 1985) to analyze all the manifestations of language in the mathematical learning process, from the language used in textbooks to that developed experientially by the students. The Siena team has made mathematical discussion (Bartolini Bussi, 1996) an object of their research, focusing on the potential for didactical use of teachers' communication strategies. The ETL group has analyzed students' talk in small group project work focusing on the social aspects of communicational intent and how that interacts with the process of learning mathematical meanings.

b) The socio-systemic cluster

Most of the research work in which the TELMA teams have been engaged is within the framework of design research. In some sense, a didactical intervention is designed and implemented and the investigation focuses on various aspects of educational practice resulting from this intervention in normal everyday practice. Since most of the work addresses educational processes within educational systems, important and influential parameters concern the socio-systemic contexts of the actors involved. These may address the organizational pragmatics of the University or the lab, the relationship between the researchers' organization and the educational sites...

Indeed design studies aiming to intervene in some way in normal educational life, inevitably cause some kind of perturbation (Laborde, 2001). This perturbation is not only at the level of the actual educational process in the classroom involving practical issues (e.g. everyday schedules and technology use management) but it also involves much deeper issues at the socio-systemic level (Jaworski, 2004) of how this intervention is materialized between organizations and in what type of organizational context this collaboration takes place (e.g. teacher-student roles, social orchestration in the classroom, epistemologies and beliefs about mathematics and the educational process). The introduction of new practices, relationships and epistemological frameworks can lead to complex and potentially contradictory positionings for the participants (Morgan, Tsatsaroni & Lerman, 2002).

One can further structure the concerns referring to the socio-systemic dimension into three sub-dimensions⁵. These are elaborated below by questions which may be used to interrogate and illuminate the socio-epistemic context of a given intervention.

The first sub-dimension seems aims to locate the designed didactical intervention with respect to "normal" everyday practice.

- What type of research is followed (e.g. classroom based, case studies) and how is it related to the kind of research focus?
- How is the lab situation/structure taken into account in the research design?

The second relates directly to the relationship between the researcher and her/his institution and the school or other institution within which the intervention is implemented.

- How are the socio-systemic factors addressed: administration, teachers in daily action, roles and relationship with researchers, daily program (time, curriculum, method)?
- What are the organizational pragmatics of the University or the lab, the relationship between the researchers' organization and the educational sites (existence or absence of institutional mechanism, e.g. part of an institutionalized pairing of University – school etc)?
- When the researchers approach a site, what is their perceived role by the administration of that site and the actors to be involved? How much personal contact do the researchers have with the actors? Does it have any effect on the research?

The last involves more directly the analysis and evaluation of the impact of the didactical intervention on "normal" everyday practice.

- What kind of 'perturbation' does the implementation of the research imply (e.g. not only practical issues but also much deeper issues like teacher-student roles, social orchestration in the classroom, epistemologies and beliefs about mathematics and the educational process)?

Let us remark that whereas earlier questions directly concern the research design and may be approached a-priori, the last one is more directly concerned with the research implementation and demands a-posteriori evaluation of the effects of such implementation.

As far as the social systemic cluster is concerned, the synthesis produced within TELMA revealed that most of the teams' approaches include some kind of participatory research at school sites where tools are studied in educational contexts (DIDIREM, ITD, NKUA). There is a common preference among the teams to approach the educational sites in the context of

⁵ This further distinction was not made within the TELMA project.

research projects. Within this framework they take the approach that the potential use of the technological tools is tightly related to the ways these will be shaped by practitioners in their respective roles in the school system (primary, secondary and tertiary). In some exceptional cases the educational activities are included in the regular curriculum (Siena, Did@TIC) while in others the research activities are implemented as part of an innovative program (NKUA) or as a frame to identify which tool functionalities are used with respect to their design to support specific learning activity (DIDIREM). An interesting finding of this overview on the teams' work is that neither the 'perturbation' implied by the implementation of research in the schools nor the relations between researchers and actors involved in the school community (administration, teachers) is a subject of research per se.

c) Teacher communication and support

The presence of the computer and in particular the presence of software inevitably represents a perturbation element in the context of the classroom. The teacher has to elaborate a new relationship to mathematical knowledge, together with the whole set of relations which link this knowledge to the use of technology in general and of specific software in particular. At the same time the teacher has to adapt his/her role of mediator taking into account the new elements offered by the software.

All of these issues involve not only teachers' time and energy but also some kind of perception of the teaching profession as a developing one and of engagement in professional development activity as a normal part of the teachers' job. Furthermore, the ways in which the intervening researchers are perceived (their official 'capacity', as well as their actual contribution to the teachers' work) highly influences the ways in which the technology will be used. All these issues can be synthesized in the following questions:

- What is the context of communication with the teachers (e.g. institutionalized channel or ad-hoc project)?
- Are there specific courses for teachers? In that case which is the frequency, duration...? How do the teachers use the course?
- What is the influence of the ways in which the intervening researchers are perceived (their official 'capacity', as well as their actual contribution to the teachers' work) on the use of technology?
- Are there indications of teachers elaborating a new relationship to mathematical knowledge, to the use of technology in general and of the specific software in particular?

As far as the teacher communication and support contextual issues are concerned, the synthesis highlights that although teachers are often taken into account explicitly in the research projects, there is very scant information about contextual issues involving their relationship with the research teams.

The issue of teachers support is addressed differently by each team. While the DIDIREM's approach looks at the teachers' professional development that would be necessary to support new instrumented practices, ITD is concerned with supporting teachers in relation to pedagogical innovation and transfer of research results. At the same time NKUA developed a multifaceted approach to work with teachers conducting research on teachers' beliefs and practices, on processes of professional development courses based on the use of exploratory software and on teachers' role in communities of practice where teachers came up with activity plans and microworlds constituting innovative educational approaches.

d) Technology design and development

Let us recall that many of the TELMA teams are both developers and users of educational mathematical tools (as well as users of professional ICT tools). Coherently the importance arises of investigating the relations and the kind of co-operation among tool-developers and educators, each having her/his own field of experience, frame of reference, epistemology and methods, as well as the phenomenon of the emergence of hybrid expertise and actors. As already mentioned this issue appears greatly underrated in mathematics education research. As diSessa puts it:

‘how people collaborate in the production of software and how software is selected and drawn into educational practice constitute a critical family of issues that are easily marginalized’ (diSessa, 2004, p.117).

The main point addressed within TELMA involves the ways in which the activities of technology design and development influence the characteristics of the tools and the nature of intervention in the school, as well as the research questions themselves.

- What is the scheme of collaboration with companies or other development institutions (on-off collaboration, discrete sequence of projects ad-hoc, long-term sustainable collaboration)?
- In case of in-house development – how is it paid for and sustained vis-à-vis persons and know how?

In the analysis of the forms of collaboration amongst developers and educators, the TELMA teams refer to diSessa’s (2004) work on the social configurations in the production of educational software leading to his definition of four distinct models:

1. the integration model, where technical and educational people combine in several small product-oriented design groups;
2. the two-legged model where collaboration between two organisations with respective know-how is maintained over a long period;
3. member-sustained community model, where the idea is ‘a symbiotic bartering of expertise’ usually found in the production of applets (diSessa, 2004, p.122);
4. the LaDDER (Layered Distributed Development of Educational Resources) model, where students, teachers, secondary developers and primary developers form for layers with an aim to empower the levels with less technological expertise.

From a synthetic point of view a multitude of perspectives and practices appears when considering the TELMA teams in relation to the activities of tool design, development and experimentation both within the TELMA project and outside. For instance the ETL team was involved in the two legged model of work collaborating with a group of technologists at the Computer Technology Institute over many years in the production of the E-slate platform.

e) Concluding remarks

As far as contextual issues are concerned, the TELMA project directly approached the practical problem of how different groups of researchers, working in different contexts, are able to understand and make use of each other's work. The main focus was centred on the issue of the diversity of the various teams' contexts in which technologies were designed, developed and then used. For that reason, the notion of context was understood in a very wide sense as evidenced by the synthetic description produced above.

The main results of TELMA work on learning contexts appear: (a) to have disclosed the complexity of the notion of learning context itself, (b) a somewhat pragmatic definition of learning contexts, and (c) the development of an analytical tool for identifying and describing contextual issues. These results provide a basis for possible further development to be carried out within the ReMath project.

Nevertheless, looking back at this work on context, some reworking seems to be necessary in order to make the notion of context more technical and operational. The introduction of many clusters and sub-clusters leads to an explosion of context dimensions, and also gives the impression that any concern can be described in terms of context. This is especially visible in part d) above devoted to the context of educational environments. We need for ReMath a notion of context better delimited. As pointed out by Otte (Otte, 2006) in the special issue of Educational Studies in Mathematics devoted to Semiotics that has just come out:

“A theory must always be simpler than the set of facts it tries to explain. Meaningfulness is thus based on abstraction and selective loss of detail [...] One of the most important aspects of systemic thinking refers to its limitations. That is, not everything can be incorporated in a given system, not everything can be explained by a given theory”

One possibility for avoiding this explosion is, following a social semiotic perspective (Morgan, 2006), to distinguish only between two main dimensions of context: the *context of situation* (for instance, for a pupil at school the immediate situation context – the classroom, the activity, etc - within he/she engages in the education experience) where an individual is immediately embedded and a more general *context of culture* which includes the former (for instance, for the pupil, but also for the teacher, this means School institution, but also the world outside the school). Let us point out that, even when introducing only these two dimensions, one can nevertheless think of context in at least two different but complementary⁶ ways:

1. as centred on subjects – individuals and communities as well – and as an enlarging sphere that progressively includes and organize in a consistent system more and more elements;
2. as centred on activities, based on the articulation, distinction and relation between different activities, for instance between the design and the possible uses of a tool (use intended in a very general sense, for instance in the school practice, or in the experimental situation)

Alternatively one can start from these two complementary aspects and try to further reduce the complexity that the notion of context addresses. In fact one could investigate the possibility of defining a somewhat “elementary context” referred to a “subject involved in an activity”, thus centring the notion of elementary context on the couple subject-activity (where subject is meant as an individual as well as a community or an institution). In the line of the TELMA work, a possible definition of “elementary context” could be given pragmatically identifying (clusters of) contextual issues, now specifically related to the couple subject-activity. Non-elementary contexts, i.e. the context of a research project, or of a team, would emerge as a result of the whole set of the elementary contexts (possibly overlapping each other) related to all the actors and the activities involved in the team, or in the research project under consideration.

The risk of such a ‘reductionist program’ is nevertheless two-folded.

⁶ Here complementarity is meant in a radical way (Sfard, 1991).

- On the one hand, as the TELMA work clearly shows, the context of the research, design and development system includes a complex system of issues; reducing the focus correspondingly simplifies the system of contextual issues – and this is the reason for reducing - but defining acceptable limits for this reduction process is not an easy task.
- On the other hand the question arises of an excessive fragmentation of macro-contexts. The juxtaposition of elementary contexts may not give account of the complexity of the macro-context: crucial issues may be lost and the articulation of elementary context too hard to be addressed (that is to say that the complexity is transferred from contexts to their articulation – inevitably entropy can only increase). This is the problem of selecting the unity of analysis (this remind us the discussion of Vygotsky on this problem and the argument that he developed about water and fire)

In TELMA, the issue of context was addressed at a pragmatic level. The work developed has evidenced the complexity of contextual issues and introduced categories for approaching this complexity. This is for ReMath an interesting starting point, but further elaboration is needed both from a theoretical and a practical perspective in order to develop an operational way of dealing with contextual issues in an integrative theoretical framework.

II.3 Theoretical frames

Since the beginning of their collaborative work, the different teams involved in TELMA were struck by the diversity of the theoretical frames they used. A better mutual understanding of these theoretical frames, of the exact role they played in their respective work on technology enhanced learning in mathematics, the search for connections and complementarities between these, thus emerged as a necessity for developing effective collaboration. As a first step in this direction, as explained before, it was decided that each team would prepare a synthetic description of the main theoretical frames it used, and would send to the other teams a reduced set of articles it considered especially insightful for understanding the type of research the team developed, the theoretical frames it relied on, and the way these influenced its work both in the areas of design and use of ILEs. This first phase of the work led to an internal report (Bottino, 2004) that, beyond the observed diversity, pointed out some general trends, and especially:

- a common sensitivity to social and cultural dimensions of learning processes, supported by different constructs, from those borrowed from activity theory or social semiotics to constructs elaborated inside the field of mathematics education itself such as those related to the theory of didactic situations and the anthropological approach;
- a common sensitivity to the ways mathematical objects are implemented into ICT tools and to the possible cognitive and didactical consequences of this implementation.

As a consequence of this sensitivity, the TELMA teams rejected the common vision of technology as a simple pedagogical adjunct and shared the conviction that ICT tools deeply affect mathematical learning both in its forms and contents. Concepts such as semiotic mediation, computer transposition of knowledge and instrumental genesis supported this sensitivity.

This first report also introduced the notion of *didactical functionality* as a means for contrasting the use of ICT tools for educational purposes and for other purposes (in the latter case, no didactical functionality is taken into account), and also distinguishing professional and educational ICT tools. This notion was defined in the following way:

“Given an ICT tool, it is possible to identify its didactical functionalities: with didactical functionalities we mean these properties (or characteristics) of a given ICT, and their modalities of employment, which may favor or enhance teaching/learning processes according to a specific educational aim”.

“The three key elements of the definition of the didactical functionality of an ICT tool are then:

1. a set of features / characteristics of the tool
2. an educational aim
3. modalities of employing the tool in a teaching/learning process referred to the chosen educational aim.”

Beyond this contrasting role, the notion was then considered as an interesting tool for anchoring the theoretical reflection in the real tasks that one has to solve when designing or analysing effective uses of ILEs (Cerulli & al., 2005), and was used for structuring the methodological tool built for investigating and comparing the role played by theoretical frames in the research work carried out by TELMA teams (Artigue, 2006). Before describing this methodological tool of direct interest for ReMath, let us summarize how the main choices underlying it were articulated in this report.

1. Taking into account that the different theoretical frames used by the TELMA teams support their research work, illuminating some important dimensions while other remain into the shade, priority was given to the areas focused on by different lights. Thus the methodological tool was designed to address common sensitivities while respecting the existing diversity in the approach to these.
2. The methodological tool was asked to support partial integrative views when these appear accessible and possibly productive, keeping in mind that a global integration is certainly out of reach, and even not desirable, the strength of any approach being attached also to the specific lens it chooses for approaching the complexity of the reality under study.
3. The methodological tool was structured around the idea of didactical functionality. This choice was seen as a means for approaching the real functioning of theoretical frames, putting these in relation with effective decisions taken as regards the use of technological tools, and trying to go beyond a declarative relationship to theoretical frames. Theoretical frames were thus questioned through the way they shape explicitly but also implicitly the vision of didactical functionalities, the means used for identifying and exploiting these, and the ways one looks at such exploitations retrospectively. The hypothesis was made that any theoretical choice conditions the vision of didactical functionalities, through the three components attached to this notion, illuminating only some facets of these.
4. To each component of the notion of didactical functionality, the methodological tool that TELMA teams built associates a set of concerns, expressed in the most neutral way, for identifying the respective areas of light and shade. Then the analysis tried to determine for each of these concerns (1) if it was addressed or not, (2) the importance given to it if addressed, (3) the associated problematization, (4) the language used and concepts mobilized, (5) the theoretical frames these expressions could be more or less directly related to, and of course, (6) the effect of these on practical decisions taken in terms of design or analysis of the educational use of ILEs.

The resulting methodological tool, structured around the three dimension of the notion of didactical functionality, is the following:

a) Tool analysis and identification of specific tool characteristics

The report points out that analysis of a tool associated to the definition of didactical functionalities generally involves two different dimensions, questioning on the one hand how the mathematical knowledge of the domain is implemented in the tool, and on the other hand the forms of didactic interaction provided by the tool. Both the implementation of the knowledge of the domain and the didactic interaction can be approached through different perspectives, which are not independent, neither mutually exclusive. The analysis and decisions resulting from the choice of specific perspectives are, among other factors, dependent on the theoretical frames referred to and on the ways these are used. These links are explored in the methodological tool through the eight following concerns:

- concerns regarding tool ergonomomy (TE)
- concerns regarding the characteristics of the implementation of mathematical objects and of the relationships between these objects (IMO),
- concerns regarding the possible actions on these objects (AMO)
- concerns regarding semiotic representations (SR)
- concerns regarding the characteristics of the possible interaction between student and mathematical knowledge (ISK)
- concerns regarding the characteristics of the possible interaction with other agents⁷ (IA)
- concerns regarding the support provided to the professional work of the teacher (teacher support : TS)
- concerns regarding institutional and/or cultural distances (ICD)

b) Educational goals and associated potential of the tool

The TELMA teams considered that it is more the relationship between potentialities and goals rather than each of these considered separately which can contribute to illuminate the role played by theoretical frames according to this dimension, complementing what is offered by the information provided by the analysis of the tool. It was thus decided to investigate the relative importance given in the definition of educative goals to considerations of an epistemological nature referring to mathematics as a domain of knowledge or as a field of practice, considerations of a cognitive nature focusing on the student in her relationship with mathematical knowledge, considerations focusing on the social dimension of learning processes, and finally institutional considerations. This led to the introduction of four concerns:

- Epistemological concerns focusing on specific mathematical contents or specific mathematical practices (E)
- Cognitive concerns focusing on specific cognitive processes, or specific cognitive difficulties (C)
- Social concerns focusing on the social construction of knowledge, on collaborative work (S)

⁷ Other agents can be the other students, the teacher, tutors as well as virtual agents such as the companions implemented in some ICT tools.

- Institutional concerns focusing on institutional expectations, or on the compatibility with the forms and contents valued by the educational institution (I)

c) Modalities of use

The design of modalities of use and the a priori analysis of their implementation supposes a multiplicity of choices of diverse nature. The TELMA teams considered it was reasonable to hypothesize that only a small part of these are under the control of theoretical frames, explicitly or even implicitly, many other being dictated consciously or unconsciously by the educational culture and the particular context within which the realization takes place. This was confirmed by the detailed analysis made of the role played by theoretical frames in the framing of the cross-experiment by DIDIREM, which was used as a first test of this structure. Seven concerns were thus selected for this third dimension:

- Concerns regarding contextual characteristics (CO)
- Concerns regarding the tasks proposed to the students including their temporal organization and progression (TA)
- Concerns regarding the functions given to the tool including the possible evolution of these (TF)
- Concerns regarding instrumental issues and instrumental genesis (IG)
- Concerns regarding the social organization, and especially the interactions between the different actors, their respective roles and responsibilities (SO)
- Concerns regarding the interaction with work with paper and pencil or other media (PP)
- Concerns regarding institutional issues and especially the relationships with curricular expectations, values and norms, the distance with usual environments (ID)

As explained above, this methodological tool was first tested by analysing the cross-experimentation design developed within the DIDIREM team. This test had positive results but showed that the three components of didactic functionalities are neither independent, nor chronologically ordered: the analysis of an ICT tool is influenced by the conjectures and anticipations one makes as regards its didactic potential and modalities of use. How to adequately take into account these interactions in the use of the methodological tool is a question which remained open at the time the report was written.

The authors also pointed out that the focus of their collaborative work was the understanding of the role played by theoretical frames in the design or analysis of uses of ICT tools, not in the design of such tools and they added that:

“Although our construction can be helpful for that purpose, other categories are certainly necessary in order to take into account the different forms of theoretical knowledge involved in the design of ICT tools and the ways these influence the decisions taken in the design process. One can also hypothesize that in design, it is a more global vision of didactical functionalities which is at stake as compared with the one used here”.

The methodological tool developed by TELMA is in an emerging state. It does not focus on representations, neither pay attention to the specificities associated to the design of ILEs. Nevertheless it constitutes a first step towards the construction of operational tools in order to grasp compatibilities and incompatibilities between theoretical frames, and organize the communication between these, and as such is of interest for ReMath.

II.4 Cross- experimentation

The cross-experimentation organised by the TELMA teams is also of interest for ReMath whose design uses a similar methodology. In what follows, after recalling the origins of this cross experiment, we describe how it was realised and analyse its first results, looking at what we can learn from it as regards the interaction between theory and practice, and the relationships between WP1, WP3 (scenarios) and WP4 (experimentations).

a) The origins of the cross experiment

The joint experiment was set up as a means for bringing forward some kind of integration among teams, while investigating on the key themes of contexts, representations, and theoretical frameworks. As explained above, TELMA teams had previously experienced the limitations due to the analysis of “finished” researches. This was the reason for adopting a new methodological tool: the cross experimentation. Such an experimentation was intended to produce a second level integration of TELMA teams, in terms of addressing a shared set of research questions derived from the three key themes of interest of the TELMA group: contexts, representations, and theoretical frameworks. As a consequence the two main issues addressed by the cross experiment were:

- investigating the roles played by theoretical frameworks, contexts and representations when concerned with ILEs in mathematics education;
- integration among the TELMA teams (at least in terms of comparing their approaches to the educational use of ILEs in mathematics).

b) The realization of the cross experiment

As previously stated, TELMA is constituted by teams which refer to different theoretical frameworks, adopt different ILE tools, employ the tools in different ways, and more generally belong to different countries, this implying different cultural perspectives and different institutional constraints when setting up research studies. In particular some of the teams had previously developed some ILE tools, and had of course experimented with them assuming theoretical frameworks, and within contexts, compatible to those taken as a reference point while developing the tools themselves. One could then ask what happens if a tool is experimented changing theoretical perspectives and contexts with respect to the assumptions of the developers of the tools. This could raise questions concerning:

- compatibility/adaptability of the tool to the new experimental context and assumptions;
- compatibility/adaptability of a theoretical framework to tool developed under different theoretical assumptions;
- differences/analogy between the two theoretical frameworks, that assumed by the designers and that assumed by the experimenters;
- differences/analogy between the two experimental contexts of reference, the one referred to by the designers and the one used by the experimenters.

It was decided to stimulate/exploit these questions within the cross experiment by putting a first strong constraint: each team had to experiment at least one tool developed by another team, possibly the tool developed by a team located in a different country. In this way, given a tool, the TELMA group attempted to experiment having the greatest distance possible between the developer team and the experimenting team, in terms of theoretical assumptions, research context and experimental context. In order to make such a choice effective, and

exploit the variety of approaches, it was decided to leave research teams free to adopt the methodologies they preferred, and to set up each single classroom experiment independently from each other. However, in order to make comparisons feasible, it was decided to reduce the variability of the experiments by imposing the following constraints:

- time constraints: the classroom experiments had to be conducted in autumn 2005 and had to last more or less one month each
- subject constraints: the experiments should concern fractions and or algebraic expressions involving fractions
- age constraints: pupils had to be between 10 and 15 years old (the age range during which the chosen subject is usually addressed in the countries involved)
- each experiment should address research questions with respect to the use of ILEs in maths education concerning:
 - representations;
 - contexts;
 - theoretical frameworks.

The last of these constraints is a very important one. In fact teams were left free to investigate whatever they wanted, so that the experiments could be fruitful not only from the perspective of TELMA, but also from each team's research perspective. Nevertheless it was required that each team had also to investigate the three key issues addressed by TELMA. In order to make the last point feasible under an integrative perspective, it was decided that the first phase of the cross experiment should be the construction of a set of guidelines to be followed by teams to set up, conduct, and analyse their specific experiments. These guidelines were built jointly according to the following procedure:

- Three researchers of the TELMA group, experts in the subjects, developed three documents (one for each of the three key themes addressed by TELMA) each consisting of a set of possible research questions to focus on.
- TELMA teams (namely the young researchers of the teams) reviewed and jointly chose a small set of questions to be addressed, taken from those contained in the three documents. The choice was done according to the following criteria:
 - relevance to teams' interests;
 - feasibility within the constraints of the cross experiment.
- Each team that produced a tool employed in the experiment was required to provide a description of the educational principles underlying the design of the tool, and to indicate possible didactical functionalities of the tool.
- Once the key questions were chosen, each team provided a plan of the experiment it was going to conduct. Such plans were included in the guidelines document in the form of answers to those of the chosen questions which could be addressed prior to the classroom experiments.
- After the classroom experiments, the teams completed the document by answering questions which could be addressed only after the experiments.

The guidelines became both a product and a tool: a product as they contain questions and answers to questions, descriptions of the experiments and results; tool in the sense that the questions and requests contained in the guidelines structured each team's work.

The analysis of the collected data is still on-going. As a first step, young researchers had to make clear the didactical functionalities of the tools that they had employed, and to try to indicate what made such didactical functionalities effective or not, and feasible or not. They are currently investigating the relationships between the teams' assumed theoretical frameworks, and the employed/defined didactical functionalities. Teams are thus required to analyse the process of design of their classroom experiments and to explain the key choices characterising such processes, and how they depend on theoretical assumptions, institutional/cultural constraints, or any other reason.

Being a multifaceted object, the cross-experiment is asked to produce different types of results. In what follows, as was announced, we synthesize those already obtained, looking at what we can learn from them as regard:

- the interaction between theoretical reflection and cases of practice;
- the relationships between the different facets of ReMath, and more especially between the elaboration of an integrated theoretical framework, of a conceptual model for scenarios, and the experimental work.

We end by pointing out some limitations of this TELMA cross-experimentation.

c) The interaction between theoretical reflection and cases of practice

The relationship between theoretical reflection and cases of practice is certainly one of the main issues that characterised the effectiveness of the cross experiment both as a tool for comparing/integrating research approaches, and as a tool for investigating how to employ ILEs in mathematics education.

In particular researchers involved in the cross experiment witnessed the importance of the request to conduct an explicit reflection on issues such as “research questions”, “theoretical frameworks”, “educational goals”, “analysis of ILE tools”, and the relationships between them, which influence each other, and which often remain implicit. The request to communicate to the other teams how each of these issues influenced/determined the design, conduct and analysis of classroom experiments, forced each team to address them explicitly, and to leave as few unexplained choices as possible. *This resulted in a very useful effort both in terms of refining each teams' investigation concerning ILEs in mathematics education, and in terms of making the descriptions of the single classroom experiments as comparable as possible.*

Moreover, when a researcher addresses such issues without being involved in a cross experiment, he/she does it on the basis of research questions formulated by himself/herself. In contrast, the TELMA cross experimentation required researchers also to address questions/issues formulated by other researchers; this obliged each researcher to cope not only with different theoretical frameworks but also with different epistemologies of research in mathematics education, possibly not compatible with his/her own. This is a very important issue, because often researchers tend to assume a defensive attitude with respect to these issues, and are rarely obliged to constructively compare each others' views. When a researcher informs other researchers about his/her work (through papers, seminars etc.) there can be a discussion on theoretical and/or epistemological issues, but it remains at the level of discussion and/or reflection. In the TELMA cross experiment on the contrary, *theoretical and epistemological differences were experienced in practice by the researchers who were required to address research questions not belonging to their own perspectives and interests, and which they may not have had the methodological background to address.*

One more important aspect has to be highlighted concerning these first results. In fact in order to make fruitful comparisons, the TELMA teams believed it was necessary to set up a collaborative atmosphere among researchers, avoiding defensive attitudes toward the cross experiment. In order to analyse finely and compare approaches to ILE in mathematics education, it was important that such approaches should be exposed in the clearest, most exhaustive, and scientifically honest ways possible. For this reason it was assumed that the different teams' constructions should be evaluated only in terms of analysing analogies and differences among them, without expressing judgements of merit. This particular choice was made possible by the fact that the focus of the communications among teams, during the cross experiments, was on factual comparisons among approaches, not on "showing how good each approach is". Let us give some examples concerning first theoretical frameworks, then contexts and representations.

Matters of priorities.

Nowadays most of the approaches to ILEs in mathematics education seem to give importance to aspects such as contexts, representations, social interactions, and the role of teachers. Nevertheless they do not address such aspects in the same ways. Such differences may remain at a reflective theoretical level, or even hidden as implicit assumptions, but the TELMA cross experiment required researchers to put in practice their views, and to compare how this was done. As a result it could witness and highlight some specific differences among the teams, increasing teams' awareness of their priorities. For instance the DIDIREM team (French culture, referring mainly to Theory of Situations and the Anthropological Theory of Didactics, etc.) gave high priority to the role of the Milieu and feedback, but during the cross experiment acknowledged a need for a specific frame for analysing and better understanding the collaborative aspects of classroom experiments. At the same time, the ITD team (Italian culture, referring mainly to socio-constructivism) gave high priority to social construction of knowledge under the guidance of the teacher, and set up the experiment focusing mainly on this issue, putting less attention on some other details of the experimental Milieu. In particular it was assumed that all the details that were left "undefined" would have been addressed by the teachers during the classroom activities. However, the classroom experiment and the comparisons with other teams, highlighted a need for a finer specification of some aspects of the Milieu that had been given low priority; for instance, the researchers acknowledge that in some occasions the tasks proposed to pupils should have been defined more carefully (in order to avoid some misunderstandings) as instead it had been done by other teams.

Matters of details and "how to" issues

In the example discussed above, we cited the ITD team, which set up its classroom experiment giving high priority to the role of the teacher in the social construction of knowledge. This position originates in the socio-constructivist approach and in Vygotskian theories to which the team members implicitly or explicitly refer; moreover giving high priority to the role of the teacher is also an important characteristics of the main trends of Italian research in mathematics education. However, when setting up the actual classroom experiment, the ITD researchers needed to know "how to" define and exploit in practice the role of the teacher. The assumed theoretical frameworks, as they had been interpreted by the researchers, provided some indications, but did not go too much into practical details; defining "how to" exploit the role of the teacher was thus a problem to be solved by the researchers. In other words, there is a gap between what is offered by a theoretical framework, and what is needed by the researchers when putting into practice the principles of the framework within a classroom experiment. Such a gap is at the core of the relationship between theoretical reflections and cases of practice, and it often remains implicit. What is interesting in the case of the TELMA cross experiment, is the fact that such a gap could be

made explicit through comparisons among the different teams' experiments. In fact, as suggested also by the examples previously discussed, each framework provides more details and practical information on some aspects than on others, thus the gaps between theory and practice are differently shaped for each framework. As a consequence, teams referring to and using different frameworks may view each other's works as a help for the identification of gaps between theoretical positions and experimental practices. In this sense it is interesting to consider the case of the DIDIREM team which is particularly familiar with addressing the roles played in learning processes by "ruptures" and "obstacles", as these are key elements of the theory of situations. During the cross experiments the DIDIREM team observed how the Siena team assumed a Vygotskian framework which describes the importance of "ruptures" and "obstacles" but which does not provide explicit methodological tools for putting this idea in practice; nevertheless, as observed by the DIDIREM team, the Siena team successfully set up an experiment where "ruptures" and "obstacles" were exploited as means for achieving a specific educational goal. The DIDIREM team expressed the will to understand how the Siena team put in practice such a principle, which started a discussion, still on-going, that is clarifying (at least partially) the gap between the Siena team's theoretical assumptions and how they put them into practice (which is certainly an original part of the team's work).

Adaptation of tools to research contexts

During the cross experiment some difficulties arose when teams attempted to use a given ILE in a context (both in the sense of school and of research context) different from that in which it had been developed. For example, the software Aplusix has been designed (by the French team MeTAH) to facilitate the teacher's work, and to offer him/her a good level of autonomy with respect to standard algebra curricular activities. The software allows students to build and transform algebraic expressions freely and to solve algebra exercises by producing their own steps as on paper; for each step the system gives an indication of correctness as feedback. Aplusix was designed to support the standard activity of algebraic manipulation, based on the solution of calculation tasks like *expand*, *factorise*, *solve the equation*, etc. However, the CNR-ITD team, adopting a socio-constructivist approach, faced the problem of planning open-ended tasks. According to this theoretical framework, open-ended tasks favour pupils' construction of meanings through exploratory activities. This was achieved by interpreting the feedback concerning the correctness of steps as feedback concerning the equivalence of expressions and/or statements. This change of perspective implied also that Aplusix was no longer used autonomously by students, but required the teacher to orchestrate the activity by asking the students to make their strategies explicit, to justify them and to discuss them with their classmates.

Representational distances as didactical tools

The NkUA team, referring to the tool ARI-LAB2 and considering the representational infrastructure of the Fraction Microworld, identified a 'distance' between the mathematical represented objects constituting the representation of fractions in the microworld (a graphical representation on the real line, allowing the user to build fractions by means of commands based on Thales theorem) and those found in the traditional primary curriculum (part-whole scheme). For example, the numerical representation of fractions with a numerator equal to 1 coincides with the part-whole representation of these fractions, which does not happen with any other type of fractions. In this case the team adopted a perspective in which it was not assumed that the object of the exercise was to minimize the 'distance' between tool design and aspects of didactic knowledge of fractions, in order to achieve use in school as quickly and smoothly as possible. On the contrary, this kind of 'distance' was considered as a challenge to provoke unexpected pupil's responses when trying to interpret the feedback provided. This decision was also reinforced by the fact that specific emphasis has been given

in the design of this tool to interaction issues and feedback offered: the software only represents fractions in specific ways and does not signal some kinds of mistakes by means of a visual feedback. This view is coherent with the NkUA team's general approach for which a tool needs to incorporate representations and functionalities at the level of artifact with which innovative didactic approaches as well as the development of different accessible strategies to solve a given problem might be supported.

Representations and institutional/cultural constraints

Adapting the way in which an ILE is used to a changed context, even if possible, may also be complicated by the role played by different curricular constraints and school praxis. As an example, we consider again ARI-LAB2. Some teams encountered difficulties using the fraction microworld in their school context due to the fact that Thales theorem is usually introduced in the curriculum later than fractions. The MeTAH team tried to use it as a "black box" but found this caused problems when pupils needed to make sense of feedback. Similarly, the DIDIREM team decided to switch to other microworlds of ARI-LAB2 because they judged it was not realistic to ask a teacher to change the mathematics organisation of the school year. However, after the first analysis and comparisons between teams' classroom experiments, the DIDIREM team went back on its decision and hypothesised that even within their scholastic context it could be possible to experiment ARI-LAB2, but under certain conditions, such as switching to long term experiments instead of short term experiments.

d) Reflection on the relationships between the work on Integrated Theoretical Frame (ITF), on scenarios, on experimentations

On the basis of the reflections and examples discussed above, the TELMA cross experiment shows that:

- The construction of an ITF should be accompanied by a cross experiment so that the two activities could nurture each other. In the TELMA experience theoretical reflection gave birth to the cross experiment, structuring it, but it was then stimulated and enriched by the reflections on the experiment itself.
- In the TELMA cross experiment scenarios (though not called with this name) have been basically provided/described in the form of answers to the research questions contained in the guidelines document, and in the form of plans to be contained in the same document. In this case the "scenarios" were to be used by researchers, which constitutes a difference from ReMath's scenarios, which are to be used also by teachers (at least). Nevertheless the TELMA experience highlighted the existence of gaps between theoretical assumptions and research practice, and indicated ways for revealing such gaps. This is significant for ReMath: if scenarios are to be used by teachers and researchers, it will be particularly important to manage the relationship between practice and theory. With this respect for instance the TELMA cross experiment teaches us that in order to uncover and clarify such gaps it may be useful to set up some form of comparison between the work of the teams.
- The gap highlighted by TELMA can probably also be identified at the level of scenarios (as described by researchers and/or teachers) by means of actual experiments.
- The TELMA cross experiment showed how relationships between theoretical frameworks and experiments, can be at least partially uncovered by means of fine analysis of the whole process of designing, developing and analysing the experiment. Such an analysis can be fruitfully conducted taking a comparative perspective, and

exploiting each team's proficiencies.

- The TELMA cross experiment also showed the effectiveness of establishing a set of concerns to be addressed, a common language, and a set of shared research questions: these steps permitted the teams to define a shared focus of research, and provided the ground for performing fruitful comparisons among experiments. This can be taken as a methodological tool for building an ITF together with a shared experimental field.

e) The limits of this cross-experimentation

The TELMA cross experiment was conducted in very restricted conditions. On the one hand, this was helpful in terms of making comparisons feasible, on the other hand it highlighted a need for revisiting such constraints in order to allow explorations toward directions that in this experience had to be abandoned. For instance, the introduction of some black boxes representations in some contexts may require long term experiments (as shown by the DIDIREM example previously discussed). In other cases, the chosen educational approaches have been explored only partially; for instance, under the socio-constructivist hypothesis assumed by the ITD team it would be very important to set up and exploit class discussion so that learning can happen in the form of social construction. However, such a pedagogical strategy, in order to be effective, needs a lot of time both for the pupils to get used to discuss and for the teacher to get used to orchestrate mathematical discussions. This of course could not be done in the time constraints of the TELMA cross experiment, so this aspect was only partially developed.

II.5 Conclusion

Coherently with what has been argued in the presentation of the ReMath project, it seems quite evident that the work developed in the frame of TELMA is of value for ReMath. The two deliverables produced on representations and contexts are situated at the core of the ReMath concerns. They are insightful in the way they connect a general and structured analysis and illustrations that help us to make sense of this analysis, and of the categories and objects this analysis relies on. The work on theoretical frames goes beyond the sole consideration of representations and contexts, situating the two first reflections into a more global but still coherent whole. Calling on the three dimensional notion of didactical functionality, it also provides us with a methodological tool for questioning and better understanding the role played by theoretical frames in the use of ILEs. This tool seems simple enough in order to be operational while considering important descriptors for linking theoretical frames and practice, and its first tests prove positive. Even if it cannot be used directly in ReMath, it can provide a useful basis for the work to be carried out. We would also point out that this theoretical work evidences two essential points we have to keep in mind when working on WP1:

- the fact that theoretical frames, be these local or more global, do not fully determine the design of situations aiming at an efficient use of ICT tools. Many decisions taken in the design of such situations as well as in their management in classrooms, once they have been designed, engage other forms of rationality or are shaped by cultural and institutional habits and constraints.
- the fact that theoretical frames themselves often act as implicit and naturalized theories, more in terms of general underlying principles than of operational constructs.

These characteristics certainly contribute to explain why the first step of the TELMA work based on the reading of published papers was only moderately productive. Making the role

played by theoretical frame visible and not just invoked needs specific methodologies. From this point of view, the part devoted to the cross experimentation evidences the interest of this methodology, and the role played by the differences in mathematics and didactic cultures of the different teams involved in TELMA in the productive character of this methodology. Moreover, the precise analysis of its outcomes and of the possible links between these outcomes and the characteristics chosen for the cross-experimentation, provides us with insightful views for connecting the work between WP1, WP2 and WP3 in ReMath.

Nevertheless, what has been achieved in TELMA, however useful it could be, does not give us immediate answers to the different problems we have to solve in ReMath. It tends to show that the metaphore of *networking* is, as regards the idea of integrative perspective, better adapted than the metaphore of *unification*, but it only suggests some hints as regards the strategies we could engage for making this networking productive. Moreover the notions of representation and above all of context have to be more operationally developed. To these limitations adds the fact that the TELMA reflection has been more focused on the design and analysis of uses of ILEs than on their design. In ReMath, design of ILEs is an important component of the project, and the extensions proposed have to increase the potential of communication of the different tools. Even if we are aware that theoretical frames do not determine design, we have to better integrate the work on theoretical frames and the work on design more fully.

Before entering a constructive phase we thus decided, as explained in the introduction, to look at complementary sources, and try to figure out if the lessons drawn from the TELMA experience were coherent with those drawn by other researchers in the field. We present this component of our work in the next part.

III. The analysis of complementary resources

As mentioned in the introduction, research in mathematics education is more and more sensitive to the difficulties induced by the diversity of the theoretical frameworks used in this field and the limited efforts made for developing integrated perspectives or at least organising some efficient communication between these frames. Beginning our integrative work in ReMath, we thus decided to look for articles, chapters or special issues of journals recently published which could nurture our theoretical and methodological reflection. We also decided to limit our search neither to texts dealing with technology nor to texts produced by the community of mathematics education as, on the one hand the reflection on the theoretical frameworks underlying research work on representations is not limited to technological contexts, and on the other hand we suspected that research carried out in the EIAH community could offer insightful and complementary perspectives on the theoretical dimension of the design of ILEs. We also decided not to develop a comprehensive study of the literature but select some few texts especially meeting our concerns. This led to the list of texts mentioned in the introduction. This work is still in progress, and new resources that have just come out such as the special issue on semiotics of the journal *Educational Studies in Mathematics* (Saenz-Ludlow & Presmeg, 2006) or the book published by Hermès on EIAH (Grandbastien & Labat, 2006), or are coming out such as the Proceedings of CERME4 will be integrated into our reflection in the next months.

In this part, we illustrate this facet of our work by using four sources quite different. The first three are reflective texts published or to be published as book chapters. They approach the existing theoretical diversity from different perspectives: mathematics education or EIAH research; the first two directly address the technological context while the third is more general; the three of them moreover try to provide some methodological grids for

investigating possible complementarities and conflicts. The fourth text is rather different as it focuses on representation issues, synthesizing the long term reflection carried out in an international group. For that reason, we treat this last text separately.

III.1 A meta-study of the literature

This meta-study was a national project, resulting from a call for research issued by the French Ministry of Education, concerned about the poor returns from research and innovation on the educational uses of computer technologies. It started as a study of a comprehensive corpus of research and innovation publications in the field of ICT integration in mathematics, published from 1993 to 1998. For that purpose, the authors used, as reported in (Lagrange & al, 2003):

“a variety of international sources (The "Zentralblatt für Didaktik der Mathematik" database with the entry "Computer Assisted Instruction", four international journals on mathematical education, seven international journals on computers for mathematics learning, books with chapters on technology and mathematics education, etc.) as well as French works (professional and research journals, dissertations, research and official reports, etc.)”.

This resulted in a corpus of 662 published works. What makes this meta-study of particular interest for us is two-fold:

- Unlike classical meta-studies, this meta-study does not focus just on research and innovation results but pays specific attention to the set of connected problems and questions addressed (what the authors call the ‘problématique’ of a study), the theoretical frames used and the ways these are used, the influence these theoretical frames and ‘problématiques’ seem to have on the methodological choices and on the results eventually obtained. This goal was especially achieved through the qualitative and quantitative analysis of a sub-corpus of 79 publications representative of the diversity of the whole corpus, and being informative enough.
- For developing a statistical analysis of this selected sub-corpus and look, through a cluster analysis, for publications especially representative of the different trends in the literature, the authors decided to identify dimensions and indicators allowing them to code the selected publications. The definition of these dimensions and indicators resulted from the qualitative analysis of the ‘problématiques’ of another sub-corpus: that of publications related to CAS (Computer Algebra Systems) technology (about 150 publications).

The resulting dimensions were the seven followings:

1. the general approach of ICT in education,
2. the epistemological and semiotic dimension,
3. the cognitive dimension,
4. the institutional dimension,
5. the instrumental dimension,
6. the situational dimension,
7. the teacher dimension.

and the associated indicators are briefly described in the table below copied from (Lagrange & al, 2003)

1. General approach of the integration	<ul style="list-style-type: none"> • Type of hypothesis (assumption of improvements, questions, etc.) • Methodology and validation processes (comparing experimental and control groups, comparing a priori analysis and expectations with an experiment, etc.)
2. The epistemological and semiotic dimension	<p>Influence of ICT</p> <ul style="list-style-type: none"> • on the mathematical knowledge and practices, • on the representatives used in this activity
3. The cognitive dimension	<ul style="list-style-type: none"> • Cognitive frame (constructivist, socio-cultural,..) • Concepts used (schemes, webbing, etc.) • Cognitive role of ICT (visualisation, expression, connection, etc.)
4. The institutional dimension	<ul style="list-style-type: none"> • Interaction of ICT with tasks and techniques in the culture of a school institution, • Role of instrumented techniques in conceptualisation of mathematics.
5. The instrumental dimension	<ul style="list-style-type: none"> • The tool's possibilities and constraints, • Instrumentation processes.
6. The situational dimension	<p>Influence of ICT on</p> <ul style="list-style-type: none"> • the structure of the situation • students' solving strategies • the didactical contract
7. The teacher dimension	<ul style="list-style-type: none"> • Teacher's beliefs and representations of mathematics and of ICT • New teaching situations • Influence of research and pre/in service programs

Table 1: Indicators of dimension in (Lagrange & al., 2003)

According to the authors, the general picture given by the cluster analysis carried out was that of “a field where publications about innovative uses or new tools and applications dominated”. Studies focused primarily on the interaction between students and technology, and through the study of this interaction, on the ways technology influences the students’ relationships with mathematics. Important attention was paid in these both to the epistemological and semiotic dimensions of ICT and to cognitive concerns, while the last four dimensions introduced by the researchers: the institutional, instrumental, situational, and teacher dimensions, were generally poorly taken into consideration. From a theoretical point of view, the predominant reference was the reference to constructivist approaches towards learning, but beyond this uniformity, the authors pointed out the multiplicity of the approaches and concepts mentioned, and the fact that, in many publications, theoretical frames seemed more invoked than operationally used. These tendencies for instance resulted in a cluster analysis along the cognitive dimension offering little coherence.

Let us have a closer look at the results of the cluster analysis made by aggregating the two first dimensions dealing with the general and semio-epistemological approaches. The statistical analysis points at two specific papers (Tall, 1993) and (Kieran & al., 1996) respectively. These are situated in two different clusters for each of the two dimensions. According to the authors, these two papers present interesting similarities:

- “they reflect on software applications and look for their benefits in the learning of algebra,
- they emphasise the potentialities of IC technology for visualising, offering multiple representations and generalising,
- they take into account the modifications that technological representations bring to mathematical notions,
- they are based on in-depth experimentation of situations built from this analysis, but without explicit link with "ordinary" mathematics teaching.”

They also present interesting differences:

- “Kieran et al. refer their analysis of the knowledge to theoretical elaborations on the teaching and learning of algebra not specific to the use of IC technologies. They use it to look at how technology changes the practice of algebra and at possible obstacles that it could bring.
- Tall is principally interested in the effects of the experimental teaching. He measures this effect by means of external comparison (results at pre- and post- tests by experimental and control groups), whilst Kieran et al. privilege the observation of students' behaviour and solving processes.
- Tall's approach starts from the potentialities of technological environments at a relatively general level. He offers experiments as illustrations of these potentialities. When difficulties occurred in their actualisation, the paper discusses how to overcome these difficulties. In contrast, Kieran et al. look at the potentialities by focusing specifically on the software environment that they developed and experimented with. Potentialities and limits of the new technologies are considered through the options they took when designing the software. Thus potentialities can be more directly questioned.”

According to the authors, these papers illustrate the common approaches and different options found in the other papers of the corpus. Moreover, some differences observed between these two papers published respectively in 1993 and 1996, are representative of the global evolution of the literature from 1993 to 1998. An analysis of the use of technology better linked to general approaches to the teaching and learning on the one hand, more interest in an internal view into the situations of use of technology on the other hand, are the two main points they mention, using another paper from the corpus by Tall and his colleagues published in 1997 for consolidating their argumentation.

As has been mentioned above, as regards the cognitive dimension, beyond the dominant reference to constructivism, a great diversity is observed. Three papers situated in three different clusters according to this dimension respectively written by (Laborde & Capponi, 1994), (Hoyles & Healy, 1997) and (Yerushalmy, 1997) are this time pointed out. According to the authors, these three papers are for instance representative of the evolution observed during this period as regards the vision of the relationship between perception and conceptualization:

“The three papers share a special sensitivity to the role played by perception in cognitive processes. This is certainly a strong specificity of research on the use of IC Technologies. The three papers certainly also differ from papers in this field, especially at the beginning of the period of our study, which are characterised by a "naive approach" where the visual potential of technology is

emphasised and offered as means for improving mathematics understanding and conceptualisation per se.

Like most texts in the corpus the three papers are situated beyond this naïve attitude and question the potentialities of technology for visualising with regard to the learning of mathematics. Laborde & Capponi show how an interaction with dynamic geometry software gives the students an understanding of visual or mechanical constraints that their construction has to conform to. This understanding is not directly a geometrical conceptualisation but it opens new ways towards this conceptualisation because the geometrical theory might be an explanation or a model of the visual properties of dynamic geometry. For Yerushalmy, the semantics of asymptotes is not just in the graphic perception of asymptotic behaviour of functions. It is in the conjunction of graphic visualisation and symbolic representations, and in the mutual interpretations that each representation can give of the other. To favour the construction of meaning, Hoyles & Healy stress the need for students to focus simultaneously on actions, visual relationships *and* symbolic representations. They design a microworld to provide a help for this.

Like these three papers most recent publications tend to present the relationships between perception and conceptualisation in dialectic ways. The cognitive power of visualisation tools and the underlying cognitive processes are a matter of investigation rather than just assumptions. For instance, accessing geometrical knowledge is no longer presented as resulting from the rejection of some perceptive apprehension of geometrical objects, but from the ability of relying efficiently both on spatial and geometrical competencies. More emphasis is put on the characteristics of problems and situations which can foster the dialectic interplay between competencies of a different nature and thus contribute to the development of geometrical expertise.”

According to the authors, these papers also illustrate the increasing sensitivity towards the contextualization of knowledge observed in the corpus, as well as the diversity of the conceptual tools used in order to approach this contextualization. Laborde and Capponi, for instance, rely on Brousseau’s theory of situations and the notion of a didactic milieu, while Healy and Hoyles rely on the notion of situated abstraction and of web of connections between such situated abstractions. More generally, as is stressed by the authors, many papers show that:

“By introducing some distance with respect to standard teaching environments and norms, research in the use of IC technologies tends to act as a window on the situated nature of knowledge and on its dependence on the particular context in which it is built and used.”

The conclusion of (Lagrange & al., 2003) written three years after the completion of this meta-study is also interesting by the evolution it points out as regards the other dimensions, as in the quotations below:

“The 1994-1998 literature appeared to restrict its analysis to potentialities of IC technology itself (easier and more varied representations, new aspects of mathematical knowledge, etc.) rather than questions raised by its insertion into the "ordinary" mathematics teaching. Despite this restriction, it provided interesting material when phenomena observed in this period could be interpreted in new dimensions, whether instrumental or institutional. As compared with today's literature (see Mariotti, to appear), we found the instrumental dimension to be in an embryonic state. [...]

The main institutional concern that we found was the difficult viability of technology in schools. This concern was however shown through very varied approaches. Papers with a pioneer spirit started from today's difficulties to motivate the use of tomorrow's technology, while others looked for reasons in more permanent characteristics of technology and of the educational institutions. The relationship between ordinary paper/pencil work and the use of technological tools was an emerging issue. To address this issue, no real theoretical elaboration was found in our corpus when today the "techniques" are seen as an important level, intermediate between tasks and conceptualisation, and this level is taken as central in the relationship between ordinary and instrumented work (Kieran, 2001). [...]

In spite of the constructivist reference of many papers, situations of integration of technology were rarely completely analysed. We found one isolated instance of the use of a "theory of situations" (Brousseau, 1997). In recent years, the interest in this theory has grown in the international literature and it could help when looking in depth into changes in the learning situations and when showing precisely what is at stake in these new situations (Sutherland & Balacheff, 1999).

In the years 1994-1998 questions about the teacher necessarily brought about more general problems with few solutions. There was a tendency to focus on teachers' development and an implicit assumption that the transfer of innovative situations of use, possibly supported by outcomes of research, would provide the teacher with sufficient material for an easy integration. Aware of the complexity of teaching and learning situations with ICT, researchers are now more cautious. Interesting research studies start from the observation of teachers struggling to integrate ICT into the real teaching (Monaghan, 2001). The study of contrasting teacher decisions helps to consider constraints inducing teachers' privileged views on the use of ICT (Kendal & Stacey, 1999)."

III.2 The Platon 1 project

This project has been carried out within the frame of a specific action of the STIC department of the CNRS in France entitled : « Fondements théoriques et méthodologiques de la conception des EIAH ». It has led to a report published in 2004 (Tchounikine & al., 2004). Our interest for this project is once more two-fold:

- The project is focused on the conception of EIAH and more precisely on associated research, and thus appears complementary of the meta-study presented above where this dimension, essential in ReMath, was poorly addressed,
- It presents both a general reflection on the nature, aims and difficulties of research about the design of EIAH, and a multidimensional structure (Platon-1) whose aim is to lead researchers to make explicit the different dimensions of their work.

EIAH are defined in this text as « artefacts informatiques qui embarquent des fonctionnalités spécifiques liées à l'objectif de susciter ou d'accompagner un apprentissage humain ». According to the authors, the conception of such artefacts raises specific difficulties. An EIAH incorporates a didactical / pedagogical intention. The first design problem is thus that of the definition and modelling of this intention, and of its connection with the specifications of the artefact. This can be achieved in different ways, from the transposition of existing didactical or pedagogical models to the elaboration of genuine models, but generally requires the coordination of several models: models for the domain knowledge, models for actions and feedback, models for mediated interaction... The theoretical frames underlying these different models have to be made explicit; their possible connections, the precise roles they are respectively given, and their impact on the design of the artefact have to be analysed. Some of these models inspire the design at a rather general level but other have to be implemented, which results in difficult problems of operationalization. Moreover, dealing with learning processes obliges us to take into consideration the individual characteristics of the learners, the evolution of their system of knowledge and the adaptation of the environment to this evolution, as well as the evaluation of the learning processes taking place through interaction with the artefact. All of these are specific difficulties which, according to the authors, make the design of EIAH especially complex.

What is also pointed out is the fact that the scientific domains involved in the design of EIAH have experienced important changes in the recent years, and that these changes directly affect EIAH research:

« Les disciplines concernées ont connu ces dernières années plusieurs bouleversements importants qui touchent directement le cœur de la recherche en EIAH et l'articulation entre les recherches en

informatique et en sciences humaines : théories de la cognition située [Clancey 97] et distribuée [Hollan & al. 00], théorie des situations didactiques [Brousseau 98], réinterprétation de la théorie de l'activité dans le cadre des travaux sur la collaboration [Engeström 87] ou encore travaux sur la communication médiatisée [Baker 03], évolution des interfaces hommes-machines ou encore des approches informatiques de la notion de « connaissance » et de « raisonnement », meilleure prise en compte par les chercheurs du domaine de l'importance fondamentale de l'analyse des usages et de leur interaction avec les processus de conception. L'EIAH, champ scientifique au croisement de ces différentes disciplines, est profondément affecté par (et constitue un terrain d'étude de choix de) ces évolutions ».

For the authors, nevertheless, the theories used, whatever they may be, do not allow the full specification of an EIAH. Design always obeys an iterative process where understanding / modelling on the one hand, designing / implementing on the other hand, strongly intertwine.

This being said, they acknowledge that the current problems met in EIAH design do not result from a too strong emphasis and leading role given to theoretical considerations. Conversely, according to them, EIAH design is generally an empirical process whose theoretical bases remain fuzzy, and they write:

« Le verrou central qui se pose actuellement est l'absence d'un corps articulé de savoirs (théories ou éléments de théories) répondant à la problématique de la conception des EIAH, c'est à dire prenant en compte les spécificités propres aux EIAH et à leur processus de conception. Si la recherche en EIAH est largement expérimentale (ne serait-ce qu'en raison de la nécessaire mais difficile nécessité de prendre en compte les usages) et la conception et l'expérimentation de prototypes successifs sont donc fondamentales, le prototypage ne peut cependant être considéré comme un palliatif au manque de fondements théoriques [...] Les enjeux de cette recherche sont d'élaborer un corps de connaissance permettant de proposer des fondements théoriques et méthodologiques à la conception et à l'analyse des EIAH et leurs usages et, d'un point de vue application, de dépasser des processus de conception fondés sur un simple « prototypage itératif » des idées. »

The authors, finally, point out the diversity of the perspectives underlying design research in that area, from research where the creation of a particular artefact is the ultimate goal to research where the artefact is seen as a means for studying a specific issue. This diversity can be a source of enrichment but also a source of misunderstanding and an obstacle to collaboration and capitalization if, as is often the case, the conceptualizations underlying the research projects remain too implicit.

These are the reasons justifying the building of the Platon-1 structure which is presented after this reflective analysis. This structure aims at supporting the analysis and understanding of research work carried out in the area of EIAH design, and does not aim at creating something normative. It has been built through a three phase design: an inductive phase based on the analysis of existing systems and works, a phase of analysis and reflection carried out at a more abstract level, a third phase where the tool has been tested in the analysis of a small number of research projects by their respective authors.

The dimensions of analysis are structured into four different groups:

- A. Dimensions related to the definition of the research project
- B. Dimensions related to the theoretical frame of the research
- C. Dimensions related to the results of the research
- D. Dimensions related to the life-cycle of the research

As shown in the schema below:

<p>[A] Definition of the research project</p> <p>[A.1] Goals of the research</p> <p>[A.2] Constraints on the building of the EIAH</p> <p>[A.3] Finalities of the computer artefact</p> <p>[A.4] Actors involved in the design</p> <p>[A.5] Social roots</p>	<p>[B] Theoretical frame of the research</p> <p>[B.1] Reference to taught knowledge</p> <p>[B.2] Invoked theories</p> <p>[B.3] Relationship between the theory and the problem addressed</p> <p>[B.4] Role of the theory in the design</p> <p>[B.5] How the theory is mobilised in the design</p> <p>[B.6] Theory of the design</p>
<p>[C] Results of the research</p> <p>[C.1] Nature of results</p> <p>[C.2] General and generic aspects of results</p> <p>[C.3] Types of validation for results</p> <p>[C.4] Analysis of results and research</p> <p>[C.5] Impact of research</p>	<p>[D] Life-cycle of the research</p> <p>[D.1] Context for the launching of research</p> <p>[D.2] History of the research</p>

Table 2: The dimensions of analysis in PLATON 1

For each dimension of analysis, a set of prototypical situations is proposed. Let us focus on the dimensions belonging to group B. According to the authors:

« Ces dimensions visent à favoriser l'explicitation du cadre théorique de la recherche : la référence au savoir enseigné, les théories ou familles de théories invoquées, la relation entre la théorie et le problème abordé, le rôle de la théorie dans la conception ou la théorie de la conception qui est utilisée. »

B1: Reference to taught knowledge

Make explicit the relation between the research carried out and taught knowledge, for instance character specific to a particular discipline / transversal competences. B1 can be linked to B2 through epistemological and didactic theories.

B2 : Invoked theories

Make explicit the theory(ies) or families of theories invoked in the research, as for instance:

- Learning theories,
- Cognitive theories,
- Domain knowledge theory,
- Interaction and communication theory,
- Human development theories (psychological or social theories),
- Didactic theory,
- Computer science theory (representation of knowledge...)

B3 : Relationship between the theory and the problem addressed

Make explicit elements about this relationship and the interactions, connections, hybridisations between different theories, if any)

B4 : Role of the theory in the conception

Make explicit this role and/or the way it is used. For instance :

- Simple evocation of a general approach (such as constructivism),
- Looking for coherence with guiding principles,
- Providing concepts directly used or transposed in the conception (for instance : transposing the Vygostkian concept of ZPD),
- Providing precepts directly used or transposed in the conception (for instance : study of a collaborative situation adopting the perspective of distributed cognition, which leads to not limiting the analysis to internal cognitive processes, adoption of the precepts of the cognitive theory ACT, which leads to choosing retroactions of the type immediate feedback),
- Providing inference or enactive mechanisms (for instance : knowledge base systems, deduction mechanisms)

B5 : How the theory is mobilised in the conception

Make explicit the type of reference to the theory which is used in the research. For instance :

- Reference, implicit or explicit,
- Direct utilisation, transposition,
- External theory or theory implemented in the computer artifact,
- Theory generating the system (the objects and principles of the system are essentially based on the notions and precepts of the theory) or theory some elements of which are used in the conception.

B6 : Theory of the design

Make explicit the theory of design which is used in the research.

On the one hand, a model of design. For instance:

- Linear design (elaborating a set of specifications then developing the product)
- Iterative design
- Participative design
- User-centred design
- Utilisation-centred design

On the other hand, the role given to the users in the design:

- Part of the design under the users' responsibility
- How the users are involved in the design (when, by whom, in what ways...).

According to the authors, the results from the first tests show the utilisability of the process of analysis which is proposed, and its efficiency for making explicit the key points of the research project. They also point out nevertheless that carrying out such an analysis resulted difficult for the researchers involved, obliging them to adopt a perspective on their research

work different from the usual ones. The fact that some dimensions were non pertinent for a specific research work created also some turbulence.

III.3 A chapter on the comparison of theoretical perspectives

This chapter by Cobb (Cobb, to appear) directly addresses the problems raised by the existence of competing and incommensurable theoretical perspectives in mathematics education. The author's aim is "to question the repeated and unfruitful attempts that have been made in mathematics education to derive instructional prescriptions directly from background theoretical perspectives". He argue "that it is instead more productive to compare and contrast various perspectives by using as a criterion the manner in which they orient and constrain the types of questions that are asked about the learning and teaching of mathematics, the nature of the phenomena that are investigated, and the forms of knowledge that are produced".

To structure the comparison, Cobb introduces two criteria that reflect some personal values and concerns. These are the followings:

- i) how a given perspective conceptualizes the individual;
- ii) the potential offered by the perspective for the understanding of learning processes and the means it provides for supporting their realization that is to say what it offers "for formulating, testing, and revising conjectured designs for supporting mathematical learning".

The first criterion concerns the nature of the phenomena that are investigated and is chosen in order to question the usual dichotomy between approaches considering activity as primarily individual or social. For him such a dichotomy is misleading 'in that it assumes that what is meant by the individual is self-evident and theory neutral". The second criterion is tightly connected with the author's personal vision of mathematics education as a design science.

Four theoretical perspectives are thus compared:

- Experimental psychology referred to as the psychological research tradition in which "the primary methods employed involve experimental and quasi-experimental designs, preferably with the random assignment of subjects",
- Cognitive psychology, with a specific focus on "theoretical approaches that seek to account for teachers' and students' inferred interpretations and understandings in terms of internal cognitive structures and processes"
- Socio-cultural theory, directly inspired from the writings of Vygotsky and Leontiev, and whose concern is with "the process by which people develop particular forms of reasoning as they participate in established cultural practices"
- Distributed cognition, which has developed in reaction to mainstream cognitive science and while incorporating some aspects of the Russian work, tends to "restrict its focus to the immediate physical, social, and symbolic actor's environment".

As stressed by Cobb, each of these perspectives has developed a particular vision of the individual, and has different affordances in terms of design. For instance, according to him, the distributed perspective, which treats classroom processes as emergent phenomena rather than already-established practices into which students are inducted, has greater potential than socio-cultural theory to inform the formulation of designs at the classroom level. The results of his analysis are synthesized in the table below copied from the chapter:

Theoretical Perspective	Characterization of the Individual	Usefulness	Limitations
Experimental Psychology	Statistically constructed collective individual	Administration of educational systems	Limited relevance to design at classroom level
Cognitive Psychology	Epistemic individual as reorganizer of activity	Specification of “big ideas” Design of instructional activities	Means of supporting learning limited to instructional tasks
Sociocultural Theory	Individual as participant in cultural practices	Designs that take account of students’ out-of-school practices Designs that take account of institutional setting of teaching and learning	Limited relevance to design at classroom level
Distributed Cognition	Individual element of a reasoning system	Design of classroom learning environments including norms, discourse, and tools	Delegitimizes cognitive analyses of specific students’ reasoning

Table 3 : Comparison of theoretical perspectives (Cobb, to appear)

For Cobb, who refers to pragmatic realism, theoretical frames co-exist and conflict. Instead of looking at encompassing theories, we should view the various co-existing perspectives as sources of ideas to be adapted to our purposes, and he adds:

“The contrasting ways in which the different perspectives characterize the individual indicate that they are incommensurable. In terms of Putnam’s (1987) pragmatic realism, adherents to the differing perspectives ask different types of questions and produce different forms of knowledge as they attempt to develop insights into different realities. I followed Putnam (1987) and Kuhn (1962, 1977) in arguing the realities that researchers investigate are conceptually relative to their particular theoretical perspectives, but rejected the claim that any of these realities is as good as any other. The approach I took is consistent with Feyerabend’s (1975) claim that we cope with incommensurability both in research and in other areas of life by drawing comparisons and contrasts in the course of which we delineate similarities and differences. Feyerabend also argued that there is no single ultimate grid for comparing theoretical perspectives, and demonstrated that they can be compared in multiple ways. The primary challenge posed by incommensurability is to develop a way of comparing and understanding different perspectives. It was for this reason that I discussed the two criteria I used in some detail.”

III.4 Comments

These three research works converge on several points and resonate with the results obtained by TELMA teams in their collaborative work. They acknowledge the theoretical diversity in the field, and the direct influence that this diversity has on the research developed here and there, and on the forms of knowledge accessed through it. They acknowledge the difficulties generated by such a situation but do not see as a solution the building of some over-arching

theoretical frame able to encompass in a coherent whole a sufficient multiplicity of existing perspectives. The incommensurability of the underlying paradigms is seen to make this enterprise a dead-end one.

What is more advocated is the development of tools allowing us to understand better how the different perspectives influence the ways the complex reality is looked at, what they consider as interesting questions, significant phenomena, acceptable explanations and results, to identify similarities and incompatibilities. Such tools are considered to be necessarily multidimensional and the first two contributions evidence the role that can be given to prototypic examples (resulting from the cluster analysis in the first case, or chosen by the researchers in the second case) in order to help general and abstract categories to become meaningful.

What can also be taken from the first example is the fact that, even when a multidimensional tool has been elaborated, recovering the required information in research productions is not necessarily easy, as these often do not make clear how their theoretical frames have really impacted the decisions taken. What we touch on here, and was also stressed in the analysis of TELMA work is the fact that theories influence research decisions but rarely determine these in a direct way.

What is also made visible by the second example and the reflective analysis carried out as the basis of Paton-1 is the increase in theoretical complexity when technological design is taken into account and the fact that even if design and utilisation are not independent, an integrative framework taking into account the two perspectives is something specific that cannot result from the mere juxtaposition of two separate tools.

The reflection presented in these different contributions is certainly insightful as regards what can be achieved in ReMath and more especially in WP1 in order to deal with theoretical diversity, but it does not provide us with more operational answers to the questions at stake. The multidimensional grids developed for looking at the existing theoretical diversity remain at a level of very general categories and criteria. In our opinion, they do not provide tools precise enough for fulfilling the goals of ReMath, all the more as the issues of representation and context on which ReMath focuses, even when mentioned are not deeply developed. This was the reason why we decided more recently to incorporate in the corpus the specific work on representations carried out by the PME Working Group on Representations. We present below a synthetic presentation and analysis of this work.

III.5 The PME Work on Representations

A Working Group of the International Group for the Psychology of Mathematics Education met at the annual conferences between 1990 and 1993 on the topic of Representations. Arising from the work of this group, two special issues of *The Journal of Mathematical Behaviour* were published in 1998 (Volume 17, issues 1 and 2). Edited by Gerald Goldin and Claude Janvier, the articles in this collection demonstrate a range of theoretical approaches to the topic. While not a systematically comprehensive review of the field, they provide a window onto some of the theoretical perspectives employed and developed within the field and some major issues of concern and empirical investigation. However, as Kaput argues in his critical review of the special issues, they are strongly oriented towards a cognitivist view of representation, based upon the dualisms of internal-external and representation-represented. Kaput suggests that this is incommensurable with what he calls the “situationist” perspective, arising from within socio-cultural theories of learning and knowing such as activity theory (as used within mathematics education by, for example, Cobb, Yackel, & McClain, 2000).

Goldin, in an account of the discussions during the meetings of the Working Group, identifies four meanings of *representation* that played a part in the discussions of the group:

- external physical embodiments which “can be seen as” embodying mathematical concepts;
- external linguistic embodiments;
- formal mathematical constructs that act as analytical tools for formalising mathematical ideas or behaviour (such as a state-space representation of a game);
- internal cognitive representations.

In addition, he identified “transitional representations”, possibly a variation on the first two types of embodiment listed above, that are used as “temporary pedagogical devices” to clarify mathematical concepts for students (1998b). These various meanings are not on the whole systematically or consistently distinguished in the literature.

A common notion used by a number of the authors in this collection is that of analogue or *isomorphism* (or, in the case of Vergnaud, homomorphism) between representations or representational systems. This may be a structure-preserving mapping:

- between the objects and relationships of a mathematical system and those of a physical or linguistic/symbolic embodiment of that system;
- between two external representational systems;
- between external and internal representational systems.

For Greer and Harel, an individual’s internal mental representation of a problem situation involves construction of an isomorphism, while for Hall the isomorphism is a theoretical analogy between a procedure carried out with concrete embodiments (such as Dienes blocks) and a symbolic procedure. This notion is used both to theorise cognitive functioning and to propose recommendations for instructional design involving the use of concrete or symbolic embodiments. Such design recommendations are made on the basis of being particularly well suited for specific purposes because of the specific affordances of the external representation or on the strength of the degree of isomorphism (e.g. Hall). On the other hand, some representations are identified as likely to lead to problems because of the ‘processing load’ involved in constructing the isomorphism (Boulton-Lewis) or differences between the mathematical space and the representational space (Mesquita). While there is a tendency to present the notion of isomorphism between representations as a formal relationship that can be analysed theoretically, there is also some recognition that relationships between external and internal representations cannot be completely determined. Presmeg in particular, drawing on semiotic theory, defines representation as “interpretive action, by a cognizing being” (p. 25). She argues that the ambiguity afforded by the metaphorical and metonymic aspects of mathematical systems of (external) representation enables students to establish personal meanings and relationships.

The existence and use of multiple (external) representational systems for the ‘same’ mathematical concept or system is also a common theme in both theoretical discussion of the role of representation in cognition and in discussion of instructional design. Operating successfully with multiple representations, by constructing or using isomorphic relationships (Goldin, 1998a) or by translating between different representational systems “without falling into contradictions” (Hitt, p. 125), is seen as an indicator of corresponding mathematical understanding. In some areas of mathematics, particular attention has been given to the role of multiple representations. For example, alternative representations of functions are discussed

by several contributors. Even argues that the characteristics of different problem situations require flexibility in the choice to use different representational systems that allow pointwise or global approaches to graphs of functions. Multiple linked representations are noted to be a key feature in the development of computer-based microworlds, which, Edwards points out, are commonly said to instantiate or model or embody mathematical concepts and structures. As well as posing the question of what it means to “embody” some domain of mathematics, she proposes that creating computer-based models provides “an opportunity to reconsider conventional representations, to design and evaluate new ones, and to think about how symbols and representations become meaningful to students” (p. 70).

Although there is general recognition that individuals will construct personal meanings from their experiences with external representational systems and that their internal representations will thus be different, the majority of the contributors to the JMB special issues adopt primarily what Edwards describes as an “objectivist” views of external representation. This view may be characterised by use of the conduit metaphor of communication: representations ‘carry’ mathematical meanings which have independent objective existence. It is thus possible to judge representations according to the transparency with which they carry the intended meanings. Problems in using particular representations are seen to arise from mismatches with students’ misconceptions or incorrect internal representations (e.g. Hitt; Janvier), or from misinterpretation of the function of a particular representation as an embodiment from which specific properties may be inferred (object) rather than as a translation into an alternative representational system of properties given in a problem statement (illustration) (Mesquita). In contrast, Edwards adopts a non-objectivist view when considering instructional design. Rather than seeking to build perfect transparent representations, her aim is to build artefacts and activities which provide “contexts within which meanings can be socially constructed” (p.70).

Elsewhere the general social and linguistic turn in mathematics education research (Lerman, 2000) has led to increased interest in socio-cultural theories of learning on the one hand and, on the other, linguistics, semiotics and discourse theory. These provide alternative theoretical notions of representation, in particular moving away from cognitivist perspectives and rejecting the conduit metaphor of communication. Rather than assuming a dualist relationship between an independently existing (external) mathematical concept/object and an internal mental representation, mathematical objects may be conceived as social acts and tools (Lerman, 2001). Words, symbols, diagrams etc. are no longer seen as embodiments of mathematical concepts but as culturally situated mediators of thought. Representation is thus conceived as a relationship between an object, an individual and (activity, including symbolic activity, within) the social world. For example, the authors writing in (Cobb, Yackel, & McClain, 2000) “reject the view that the process of constructing meaning for symbols involves associating them with separate, self-contained referents” and “argue that the ways that symbols are used and the meanings they come to have are mutually constitutive and emerge together” (Cobb, 2000, p. 18). Indeed, Cobb suggests that the word *symbolizing* is more useful than *representation* to indicate the activity of using symbols and making meanings.

Similarly, recent interest in using semiotic theories to provide frameworks for understanding both the nature of mathematics and teaching and learning mathematics has also led to a focus on activity and processes of meaning making: “we consider (semiotic) activity and critical awareness rather than mental representation as the central notion of epistemology” (Otte, 2006). The recent special issue of *Educational Studies in Mathematics* brings together papers by contributors to the Discussion Group on the topic of Semiotics that met at the annual conferences of the International Group for the Psychology of Mathematics Education in 2001

and 2002. As announced above, we will analyse it in a systematic way in the second phase of our work.

IV A first structure for an integrative theoretical framework

After synthesizing these preliminaries studies, we come now to the presentation of a first structure for an integrative theoretical framework. We first try to make clear what we expect from such an integrative frame, and why. We then present the main choices made in the elaboration of this structure with the rationale for these, and the resulting structure.

Before entering into more details we would like to clarify our position as regards theoretical frameworks and more generally the role to be given to theories and to theoretical development. The current proliferation of theoretical constructs --as if any valuable piece of research should contribute to the development of the theoretical edifice-- makes this clarification necessary. Theories certainly play an essential role in research, and understanding better this role in order to improve communication and capitalization of knowledge is essential but developing theories is not the *alpha* and *omega* of research. Building an integrative theoretical frame and testing it through the systematic questioning of the ReMath elaborations creates the risk of over-estimating the role played by theories in the work developed by the different teams, and also of over-valuing this dimension of research work. We will have to be aware of this risk when negotiating the use of the ITF and limit its possible effects.

Moreover, building an integrative framework, we are obliged to consider the diversity of concerns that the theoretical approaches used try to grasp. This could support the fallacious idea that theoretical approaches can or must be able to grasp everything. In contrast with this position, we consider that choosing a theoretical frame is choosing a coherent but necessarily limited perspective on the complex reality under study, and that the power of a theory also results from the limitations that such a choice imposes.

In some sense, theories have to obey a “mini-max” principle: through a minimal number of constructs, they must offer the maximum potential for identifying or producing interesting phenomena within the considered perspective, for understanding these phenomena and the rationale for these, for organising these in systems and structures. Once more, we will have to make this clear when using the ITF in order to avoid misunderstandings and biased answers.

IV.1 The aims of this integrative theoretical frame

The preliminary studies summarized in the three first parts lead us to consider again the questions raised in the introduction to this text:

- Does it make sense to look for a unified perspective, an overarching theory or meta-theory encompassing the different existing theoretical frames?
- Or is such a perspective unreasonable, due to the incommensurability of most of the existing theoretical frames, and it can only make sense to look for structures and languages allowing us better to understand the characteristics of the corresponding approaches, organize the communication between these, and benefit from their respective affordances?

The answers we propose, at this stage of our work, are respectively: No and Yes. The integrative theoretical frame (ITF in the following) we have thus in mind is neither a theory more, nor a meta-structure integrating the seven main theoretical frames used in ReMath into a unified whole. It is more a meta-language allowing the communication between these, a

better understanding of the specific coherence underlying each theoretical framework, pointing out overlapping or complementary interests as well as possible conflicts, connecting constructs which, in different frameworks are asked to play similar or close roles or functions.

Moreover, we want to seriously take into account the point that ReMath functions on the basis of a dialectic interplay between the different dimensions of the project, and through a process of cyclic iteration. This impels us to think of the first ITF as something open and flexible enough to favour its evolution through the planned cyclic iteration.

Finally, we would stress that the ITF is planned to make sense and become an efficient tool for a wide community of researchers, designers and teachers. The structure and the language have thus to be understandable by a wide range of potential users, even if we propose to include in it, at a later stage, illustrative prototypical examples to facilitate communication.

IV.2 The structure elaborated for the integrative theoretical frame

Taking into account the lessons drawn from the analysis of the TELMA work and the complementary sources, we propose to keep for the ITF the three dimensional structure around the notion of *didactical functionality*, and the language of *concerns* which seems to be effective. But considering the specific focus of ReMath on representations and contexts, we propose to reorganize the presentation of concerns around these two focuses at each level of the structure.

As mentioned above, we consider that a tool, be it theoretical or methodological, in order to be operational has to cut drastically in the complexity of the reality we look at, as if it was asked to satisfy some mini-max principle. This is the reason why we propose to limit the analysis of representations to that of external representations and consider these according to two dimensions:

- representation of objects;
- representation of interaction.

In a parallel way, we propose to introduce only two levels for contexts and distinguish between:

- a local or situational context
- a global or institutional and cultural context

This does not mean that in building a scenario or an experiment, or planning the design of an ILE, one does not take into consideration a multiplicity of factors that can be expressed in terms of contexts. But we have to keep in mind that the ITF is not a guideline serving the design of ILEs or of uses of ILEs; such guidelines are and will be defined elsewhere in the ReMath project, of course in coherence with the ITF. The ITF is a tool serving the visibility, explicitation and understanding of the role played by theoretical frames in design and analysis of design, a tool also serving a better communication between theoretical frames, and a better interplay between complementary perspectives when such interplay seems to be potentially productive. We also do not forget that theoretical frames, as has been stressed several times above, only partially determine design, and that they can influence it in different ways: providing metaphors, general principles and backgrounds, or providing operational constructs and tools.

Another way in which we address the issues of context in parallel to this two-level scheme is the very nature of the structure of ITF which is commensurable with what has come up in a bottom-up style of work through TELMA. That is, the distinction between a) what is behind a

team's production, i.e. how ILE's are designed, b) the ways in which educational environments are generated, i.e. how ILE uses are designed and c) ways in which the educational processes are studied, something which will emerge later through the experimental work.

It therefore seems to us interesting, while conserving parallelism as much as possible, to differentiate the two types of design at stake in ReMath: the design of ILEs on the one hand, and the design of uses of these on the other hand. Our work through TELMA and the analysis of complementary resources has shown why this distinction is relevant and necessary. For instance, it seems reasonable to think that, even if we consider the same global three dimensional structure for the design of ILEs, in this category of design, the first dimension: "characteristics of the tool", is often the crucial one and the part mostly impacted by theoretical frames in an operational way. Modalities of use can remain rather fuzzy for the designers or situate at very general levels of description where theoretical frames are only used at the level of metaphors and general principles.

As has been evidenced by the analysis of TELMA work, the teams involved in ReMath share common views about technology enhanced mathematics learning, and the role played by representations and contexts. Thus they are more or less sensitive to the different concerns we can attach to these. What differentiate them is more the intensity in focus they attribute to these different concerns and the specific way they approach these. For that reason, the ITF structure introduces a criteria of intensity in terms of grading from 0 to 5 for the different concerns introduced. It also takes into account the fact that one can be sensitive to a concern without engaging a theoretical approach for expressing this sensitivity. This led us to separate sensitivity to concerns from the enquiry on the role played by theoretical frames in the expression of this sensitivity. This separation can also be seen as a way for limiting the theoretical slant mentioned above.

Thus the following ITF structure where the different concerns are voluntarily expressed in a very synthetic way, the terms used, we hope, making sense for the reader who has read the previous parts of the report. The first part deals with the global contextual characteristics of the project under study, which can deal with the design of an ILE or the extension of a given ILE as will be often the case in ReMath development, with the design of use for an ILE or a set of ILEs, or with the analysis of uses of ILEs. The second part deals with design and is structured around the expression of didactical functionalities with a specific focus on representations and contexts. The third part is concerned with the role played by theoretical frameworks in the effective analysis of uses. As explained above, the structure is built in order to make visible to what concerns researchers are most sensitive and up to what point this sensitivity is supported by theoretical constructs and approaches. It distinguishes between a metaphoric and operational use of theories, this distinction being considered important when thinking of complementarities and connections. The structure can be used for different kinds of projects involving design and/or use of ILEs as explained above.

Integrative Theoretical Framework

Part 1 : Contextual characteristics of the project under study

How are the following dimensions of context taken into consideration at a theoretical level in the project? What constructs are used for this purpose ?

- The situational context of the project
- The institutional/cultural context of the project

Part 2 : Didactical functionalities and design

For each dimension of didactical functionalities, a list of concerns is given. You are asked to grade them from 0 to 5, this grade reflecting the level of priority given in design (0 not considered, 5 high priority). In a second phase, you are asked to say what are the theoretical frames you use, if any, when taking into account these concerns, and how you use these. Both representations and contexts are considered.

a) Characteristics of the ILE (or of the set of ILEs if several ILEs are concerned by design)

Are the following concerns given a high priority in your design (grade from 0 to 5: 0 not considered, 5 high priority):

- concerns about the ways mathematical objects and their interaction are represented?
- concerns about the ways didactic interactions are represented?
- concerns about the ways representations can be acted on?
- concerns about possible interactions, connections with other semiotic systems, including the representations provided by other DDAs?
- concerns about the relationships with institutional or cultural systems of representation?
- concerns about the rigidity/evolutive characteristics of representations?

For those considered, what are the theoretical frames and constructs, if any, which you refer to:

- at the level of general principles and metaphors?
- at an operational level?

b) Educational goals

When thinking about educational goals to be associated to the ILE or set of ILEs, in the design phase, what concerns are given a high priority (grade from 0 to 5):

- epistemological concerns?
- semiotic concerns?
- cognitive concerns?
- social concerns?
- cultural and institutional concerns?

Up to what point are those considered linked to representational characteristics of the ILE or set of ILEs (grade from 0 to 5: 0 no link, 5 strong link) ?

For those linked, what are the theoretical frames and constructs, if any, used for this linkage:

- at the level of general principles and metaphors?
- at an operational level?

Up to what point contextual concerns shape the vision of educational goals here (grade from 0

to 5: 0 does not shape, 5 strongly shapes):

- local concerns?
- global concerns?

What are the theoretical frames and constructs, if any, used for that:

- at the level of general principles and metaphors?
- at an operational level?

c) Modalities of use

When thinking about possible modalities of use in the design of this ILE or set of ILEs, what concerns were given a high priority (grade from 0 to 5):

- concerns about the mathematical tasks and their temporal organization ?
- concerns about the functions to be given to the artefact and their possible evolution ?
- concerns about semiotic issues?
- concerns about instrumentation processes?
- concerns about social organization and interactions?
- institutional and cultural concerns?

Up to what point are those considered linked to representational characteristics of the ILE (grade from 0 to 5) ?

For those linked, what are the theoretical frames and constructs, if any, used for this linkage

- at the level of general principles and metaphors?
- at an operational level?

Up to what point contextual concerns shape the vision of modalities of use (grade from 0 to 5):

- local concerns?
- global concerns?

What are the theoretical frames and constructs, if any, used for that:

- at the level of general principles and metaphors?
- at an operational level?

Part 3 : Analysis of use

Collection of data

How do concerns about representations and contexts are taken into account in the collection of data as regards the use of ILEs?

What are the theoretical frames and constructs, if any, used for this:

- at the level of general principles and metaphors?
- at an operational level?

Analysis of data

How do concerns about representations and contexts are taken into account in the analysis of

data as regards the use of ILEs?

What are the theoretical frames and constructs, if any, used for this:

- at the level of general principles and metaphors?
- at an operational level?

Table 4: Structure for the Integrative Theoretical Frame

Practically, we propose to associate different graphs and tables to this ITF in order to provide information in a format appropriate for looking at similarities and differences. For instance, radar charts can be used for visualizing the respective priority given to the different concerns. Tables can be used for synthesizing the information given about the theoretical frames and constructs involved. We give some examples of these below.

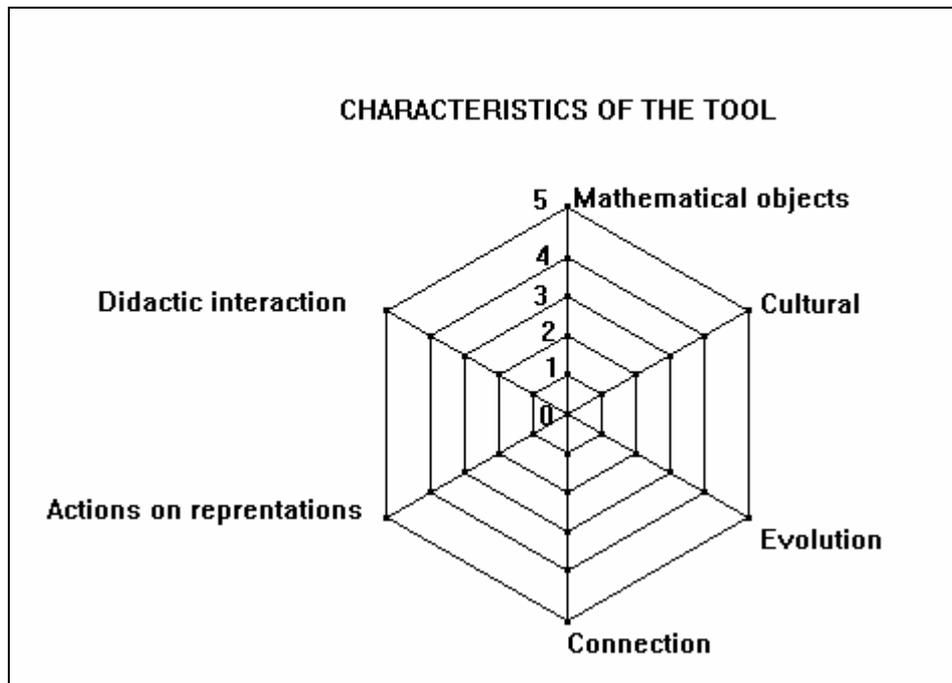
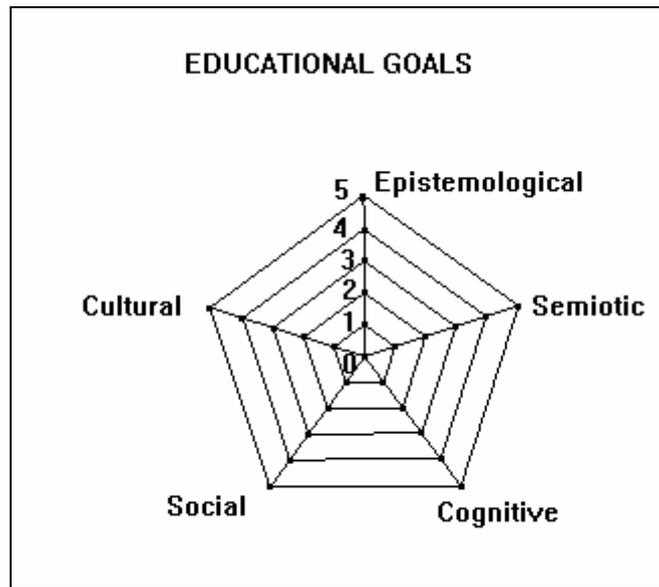


Figure 1 : Radar charts associated to the ITF

Concerns regarding:	Grade	Main theoretical frames referred to	Principles	Operational constructs
Representation of mathematics objects				
Representation of didactic interaction				
Possible actions on representations				
Connections between representations				
Relationships with cultural representations				
Possible evolution of representations				

Context dimension	Grade	Main theoretical frames referred to	Principles	Operational constructs
Local/Situational				
Global/Cultural				

Table 5: Tables associated to the ITF

Having developed this first structure, it is now important to explicit the way it will be used for connecting the ReMath work in the different workpackages. As announced at the beginning of this text, we decided in the first six months of the project to focus more on the connection with two other dimensions of ReMath : the elaboration of a conceptual model for scenarios and the cross-experimentation. We develop the way this connection is seen today in the next part.

V. Relationships between the definition of an integrative theoretical frame and the other dimensions of ReMath

As announced, we focus here on the connection with WP3 and WP4, but nevertheless introduce some elements as regards the connection with other WPs.

V.1 Integrative theoretical framework and the development and/or extension of DDAs (WP2)

As mentioned above, the connection between the development of ITF and the extension of DDAs has not been a priority up to now. A first step towards this connection nevertheless was launched at the first meeting of ReMath, in December 2005: each team involved in the design of DDAs was asked to provide a description of its DDA, making clear the key mathematical concepts for it, the representations for these implemented in the DDA, the choices made in the design and development phase as regards these and the rationale for these choices. We began

to summarize the answers collected, tried to identify basic choices in the development of the DDAs, the rationales given for these and determine if these choices and rationales were linked in the description with specific theoretical notions. We present below an example of analysis made from the description provided for the DDA MachineLab. It shows the kind of information we collected and what could be inferred from it. We then introduce the projects we have now as regards the connection with WP2, where we try to overcome the evident limits of this first attempt and also benefit from the work on specifications developed in WP2 during the first six months of the project.

a) Analysis of the MachineLab description

A microworld

We found first that the MachineLab developers choose to build a programmable exploratory and constructionist software (microworld). They do not provide for specific rationales motivating this choice and do not refer to more specific theoretical notions. Certainly they think that this broad choice is common among math educationist so that it does not need more specification. In contrast, rationales are provided for two more specific choices. One is about curricular issues and the other one is about the structure of MachineLab as a software application and about the activity that a learner can develop by interacting with it.

Integrating distinct curriculum subjects

MachineLab's mathematical central object is the notion of vector. The MachineLab developers see this domain as obscure in the mathematics curriculum, or fragmented in different sections. They explicit their goal: making students generate meaning around the notion of vector and helping to integrate distinct curriculum subjects. This goal motivates the extension planned of existing representations and functionalities provided by the current version of MachineLab from the 2D to the 3D spaces. More precisely they want to provide new means for linking 2D and 3D representations. The rationale is that they see this link as an important constituent for the understanding of geometry and motion in space and that "vectors can be considered as basic components underpinning the study" of these domains. They introduce a specific theoretical idea to give account of their goals: 'vectors-in-use' and 'vectors-under-construction'.

The MachineLab's structure and the learner's activity

The team sees MachineLab as a hybrid between a symbolic programming environment (such as Logo) and a dynamic manipulation system (such as Dynamic Geometry Environments). The programming language is a functional procedural and recursive language with manifest data types, and its syntax is aligned with the syntax of mathematical formalism. Any turtle heading and motion represents a vector. The difference between two consecutive vector inclinations represents the curvature (the differential) of the trajectory as the length of the vector tends towards zero.

The dynamic manipulation system offers dynamic manipulation of mathematical variation, multiple representations of 3D phenomena and associated feedback.

Regarding the learner's activity, this design is intended:

- to favor students' actions with the artefacts and verbal communication while doing so;
- to facilitate multiple didactical decisions within open-ended exploratory tasks;
- to allow diversity in individual approaches.

More precisely, the coexistence of the idea of pre-fabricated generic black box software and

functionalities, and white box programmability should afford teachers and students the option of constructing artefacts for themselves or for others to use.

This construction should help conceptualisation by:

- helping students re-build mathematical structures for themselves;
- favoring interaction “between the intuitive, the formal and the procedural aspects concerning vectors”;
- interconnecting the construction of personal meanings and the standard mathematical discourse.

The MachineLab team characterizes the designed functionalities by the specific theoretical idea of ‘half-baked’ software, that is to say by providing users with deep structural access to technology and taking its meanings in use when operationalised in activity.

When asked to describe the role that theoretical frameworks have played in these decisions the MachineLab team says that it wants to synthesize constructionist (Harel & Papert, 1991, Kafai & Resnick, 1996) and socio-cultural approaches (Cobb & Bowers, 1999).

The team also links its expectation that neither the students nor the artefact will remain unchanged after a mathematical activity with MachineLab has taken place, with the *instrumental genesis* perspective: in each phase, someone designs an artefact for someone else to use and changes his/her own understanding in the process of doing so.

This analysis makes clear the great sensitivity the team has as regards different concerns listed in the ITF, in particular concerns about the ways mathematical objects and their interaction are represented, about the ways representations can be acted on, about the possible evolution of representations and functions of these through programming activities, and about the social dimension of learning processes. From a theoretical point of view, this sensitivity is mainly supported by the ‘microworld culture’ and socio-cultural approaches such as those developed by Cobb. But the description does not make clear if these are used at a rather general level or at an operational one, and if so what theoretical constructs have been more especially engaged in the design and development of MachineLab.

b) Plans for future connections

During the first six months of the project, the work on WP2 has been devoted to the definition of precise specifications for the planned extension or development of the different DDAs. We consider that this work and the associated deliverable can be a very good basis for developing further the connection between WP1 and WP2. For making this connection operational, and going beyond the kind of description provided by the first methodology, we propose to have in each of the teams engaged in development one researcher more specifically in charge of this connection. These researchers could carry semi-structured interviews of the designers, on the basis on a guideline interview inspired by the ITF commonly elaborated between the different connecting researchers and the researchers in charge of the two workpackages. The data collected would then be used, in a bottom-up process in order to improve the ITF structure as regards its design of ILE component.

V.2. Integrative theoretical frame and the concept of scenario (WP3)

a) General considerations

Within the project, the ITF has got connections with the concept of ‘scenario’ which is under development in WP3. Scenarios can be seen as representations of the ‘educational activity

plans' involving the use of the different DDAs produced by the partners and include the design of the teaching/learning process in its complexity (including the theoretical framework underpinning its design and the way in which activities are to be carried out), so to foster reflections and discussions about the innovative aspects enhanced by the use of technology itself.

Within the on-going development of the conceptual model of scenario, a concerted effort has been made to be in line with the work undertaken on the theoretical framework in WP1 and with the general approach adopted in ReMath as a whole. The conceptual model of scenario does not reflect any particular theoretical framework. Rather, it aims to provide a basis for referring to a range of different conceptual frameworks through a number of common attributes linked to the concept of didactical functionality. Specifically, while the ITF is based on a set of concerns for special attention (see section IV), the conceptual model of scenario is based on a set of attributes aimed at describing the design of an actual teaching/learning process to be enacted. Even if there is no one-to-one correspondence between ITF concerns and scenario attributes, they are both coherent with the general perspective adopted in ReMath in that they seek to be as free as possible from dependence on local theoretical assumptions. In particular, the conceptual model of scenario aims to allow researchers of different teams to design/describe scenarios assuming (and/or referring to) different theoretical frameworks with the sole constraint of using a shared set of attributes to describe their scenarios. In this model of scenario, each attribute can be seen as a "container" which can be "filled" by researchers in the ways they see fit. Thus, when a scenario is designed/described, the relevant theoretical assumptions can be included in the researcher's description provided for each attribute. In other words, the conceptual model of scenario does place constraints on the issues (the attributes) to focus on when designing/describing a scenario, but allows flexible choice and use of the theoretical assumptions.

b) Supporting communication among researchers

Thanks to the considerations made above, scenarios may foster a certain level of communication and exchange among researchers and – from this perspective – a significant contribution will derive from their use in WP4 (cross-experimentation). During WP4, each team will be asked to prepare two class experiments, one involving the use of its own DDA, and the other one involving the use of a DDA designed by another team. In this phase of the project, scenarios will be used as a common tool for research teams to discuss and analyze the experiments. They may support comparison between, for example, learning experiences based on use of the same DDA but underpinned by different theoretical frameworks or, conversely, that share a common theoretical basis but instantiate these through the use of different DDAs.

In their activities, the researchers involved in ReMath have come to appreciate the importance of explicit reflection on all areas of field experiments, including such aspects as theoretical frameworks, multi-faceted goals, context and setting, specific work plan, and the relationships between these aspects – elements which often remain implicit. Scenarios represent a tool for shedding light on how such aspects (and the specific attributes used to portray them) influenced/determined the design, conduct and analysis of classroom experiments. Moreover, scenarios oblige each team to address these elements explicitly, and to leave as few unexplained choices as possible.

As well as affording opportunities for comparison, scenarios will also allow for the gathering of post-enactment feedback: as well as helping to improve and enrich enactment processes, this should also provide valuable research data regarding the DDAs.

c) Supporting communication with teachers

Another problem addressed by ReMath is that of bridging the gap that all too often separates the work of researchers and that of teachers; this gap distancing experimental settings from ordinary classroom contexts obstructs the valuable flow of communication between the two and stifles the potential for cross-fertilization. As indicated in the introduction to this deliverable, ReMath seeks to address this problem as well and, from this perspective, the conceptual model of scenario provides clear indications and support to help teachers adopt these practices in an effective manner within their particular context as they strive to meet specific educational goals. A teacher accessing a scenario will not only find out how best to use specific DDAs, but will also be able to grasp their key theoretical and educational principles and understand how to instantiate them within a specific context of use. The assumption here is that having to adopt a shared language for planning and representing teaching/learning processes will help to reduce the distance between teachers and researchers, and will lead to a better reciprocal appreciation and understanding of their respective perspectives and concerns.

In this sense the conceptual model of scenario aims to be a tool for representing a teaching and learning process to be enacted, taking into account the concerns both of researchers and teachers.

In conclusion, within the REMATH project, the ITF and the model of scenario can be seen as complementary conceptual tools aimed at bridging the gap between different research contexts and between research and ordinary school contexts, in that they provide a common structure for describing and comparing different approaches and perspectives in the field of mathematics education. As was the case in the first six months, they will be developed in close coordination.

V.3. Integrative theoretical framework and the organization of experimentations (WP4)

a) General considerations

The ReMath project involves the design and implementation of empirical research which entails:

- the design of teaching experiments based on the six different technologies that are developed in the WP2 and on the different “learning scenarios” constructed in the WP3;
- implementation of the experiments in real “learning” situations (regular classrooms but also informal settings of learning, such as teachers training);
- thick and systematic collection of data and analysis of the collected data.

Experimentations are expected to offer the project new perspectives at very general level providing new insight on a) means of using technologies to support learning b) design issues concerning both DDA and scenarios and c) learning processes in relation to the use of technologies. In particular, they are expected to provide the validation of each DDA both in respect to its functioning as didactical tool (for instance its effectiveness in respect to specific didactic objectives) and the consistency of such functioning in relation to the theoretical assumptions underlying its design and use. The ITF produced in WP1 will be a methodological tool for such an enterprise and, conversely, the experimentation carried out within WP4 is meant to validate the effectiveness of the ITF and possibly to provide elements for refining the ITF itself.

Besides specific experiments concerning each DDA, a Cross-Experimentation will be carry out. Each partner involved in the research activity will conduct teaching experiments centred on the use of at least two different tools, thus at least of one DDA it did not produce. Such experiments will be called in the following "Local Experiments". These will be coordinated within the Cross Experimentation process. Previous experience with cross-experimentations (see section II.4 in this document) showed its effectiveness and encourage to follow the same track; in spite of its clearly showed complexity, that work pointed out the basic need of making explicit the theoretical assumptions underlying and guiding (a) the design of the DDAs, (b) the construction of learning scenarios and (c) the implementation of scenarios in learning environments. This previous experience also showed the interest of using the construct of didactical functionality for guiding the organization and analysis of cross-experimentations. Consistently with this perspective, the construct of didactical functionality (DF in the following), which already structures the ITF, will be also given a key role both at the level of the Local Experiments and at the level of their coordination in the Cross Experimentation.

Moreover the spirit of the design of DDAs according to which the scenarios are developed should be disclosed; this allows checking the consistency of their utilization according to the original aims, making explicit the discrepancies (intentional or not intentional). At the same time, besides assuring the possibility of comparing findings coming from different studies, this frame should help to identify key elements characterizing the process of transfer from one context to another.

b) Outline of the experimental design

In this Cross Experimentation, as explained above, all the partners will be involved in experimenting tools with which they are not familiar and possibly, they will be confronted with different theoretical assumptions supporting the consistency of the experiment.

Given that the Cross Experimentation aims at enhancing our understanding of meaning-making through representing with digital media and in particular that it is meant to provide validation (a) of the functioning of each DDA as didactical tool, (b) of the consistency of such functioning in relation to the theoretical assumptions underlying its design and use, (c) of the effectiveness of the ITF and (d) to contribute to the development of the ITF itself, the precise research questions and methodology of the cross-experimentation will be jointly constructed. As announced above, this construction will be structured around the notion of DF. The identification of the precise research questions and their articulation with respect to the notion of DF may be considered a common objective of WP1 and WP4.

As far as research methodology is concerned, two levels can be distinguished in the organization of the experimentation. In fact the Cross Experimentation (top level) is articulated in and is meant to integrate the Local Experimentations (bottom level) carried out by the single teams according to a common methodology.

Although the details will result from the collaborative work in the next months, (for both levels of experimentation) the stated objectives induce to consider a methodology consistent with the general paradigm of *design-based research* (Cobb et al., 2003) and partly with that of *research for innovation* (Arzarello & Bartolini Bussi, 1998). In fact both paradigms attempt to bridge theoretical research and educational practice and address (even if with some differences) the development of theories and the conducting of teaching experiments as complementary processes: "theory and practice are generated together" (Arzarello & Bartolini Bussi, 1998, p.249).

More in detail, many features of the *design-based experiments paradigm* fit well the issues addressed by the ReMath project and the objectives of WP4 in particular. Firstly, one of their main purpose is “to develop a class of theories about both the process of learning and the means that are designed to support that learning” (Cobb & al. 2003, p.10). Secondly, they are meant to address the whole *learning ecology* (ibidem), i.e. that complex and interacting system of factors influencing learning such as classroom activities, tasks, problems, social norms, tools used, teachers' actions and so on. Finally design experiments, being conjecture driven, may result in iterative cycles of construction, implementation, validation, refinement and reconstruction of theoretical framework.

c) On data analysis

The implementation of the experiments and the possible effects will be documented through the systematic collection of different kinds of qualitative data: students' productions, videotapes of classroom interactions, individual interviews...

The methodology of Cross Experimentation encompasses the comparison of the findings of the different Local Experiments. In order to foster such comparison, criteria and types of evidence for the data analysis should be made as explicit as possible. We share Cobb et al.'s view that in the process of consensus building around the interpretations of data, the diversity of expertise and backgrounds among different researchers of the same team and among different teams is a valuable element. This comparison should result in the production of common results, overcoming the simple collection of single contributions.

To conclude we remark once again that this Cross Experimentation methodology, although highly demanding in terms of collaborative work, is expected to be highly rewarding in terms of feedback:

- on specific theoretical frameworks and their integration in a common systemic network as envisioned by the ITF.
- on design issues, both specific to the tools involved and general.

VI. Perspectives

The work carried out so far for WP1 has developed according to complementary dimensions. Most emphasis has been given during this first phase to the collection and analysis of existing resources for nourishing the reflection about what could be expected from an integrative theoretical framework and how the resulting positions could be implemented in practice. This first phase of the work strongly relied on the reflective analysis of the work carried out in TELMA during the last two years. It also relied on some complementary resources, taking into account the increasing sensitivity of research to such issues both in mathematics education and in EIAH research. These different sources led to the building of a first version of the ITF. This frame presents as a multidimensional structure organized around the notion of didactical functionality and concerns to which researchers are more or less sensitive. It introduces two main kinds of representations, two level of contexts, and distinguishes two main ways of using theoretical frames, at the level of general principles and metaphors and at a more operational level.

At this stage, we see one priority in the work on WP1:

- Establishing precise protocols for connections with the other workpackages. The work of connection has already begun, especially with WP3 and WP4, with the aim of ensuring that our respective reflections were developing in coherent ways. It can

today, thanks to the existence of the first version of the ITF, take a more operational form.

Simultaneously, we propose to progress inside WP1 in two ways:

- Going-on in the collection and systematic analysis of research work devoted to issues of comparison, networking or integration of theoretical frames on the one hand, on representations on the other hand
- Illustrating the categories of the ITF by some vignettes, using for that the synthetic work developed for this deliverable, to make these categories more meaningful and prepare broader communication outside the community of ReMath partners. This development, of course, will be done in close connection with WP5.

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