Introducing pupils to algebra as a theory: L'Algebrista as an instrument of semiotic mediation

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1. Introduction

In the last decades, a quick evolution of technologies has been bringing big changes in our daily life. Our means of interaction with the external world and our means of social interaction keep evolving bringing us new possibilities and new questions. Such trend has obviously influenced also school practice and research in mathematics education, raising questions that once simply one wouldn't have thought of. For instance, the main aim of Italian primary schools, some decades ago, was expressed by the phrase "saper leggere e far di conto", that can be translated as "to be able to read and to do calculations", and which reflected an unquestionable social need of alphabetization. Nowadays, hand calculators, changed so much our daily life that one may be tempted to consider mental and hand calculations as obsolete, thus one may be tempted to question related educational aims, questioning the "far di conto". In fact one may ask, is it still worth that pupils learn to do calculations, knowing that they will hardly ever use such an ability? The most natural answer would simply be "not, it is not worth it". Yet, if this is the answer, how comes that no educator, no mathematician, would ever think of eliminating calculations from school curricula? My personal opinion is that we have to separate the plane of usefulness in daily life, from the plane of usefulness from a broader educational point of view. In fact, even if mental and hand calculations are no more needed in daily life, they are still a founding element of mathematical thinking, and one simply can't ignore them as far as mathematical educational aims are concerned. A possible answer to the question "is it worth that pupils learn calculations?", is yes, for educational purposes, because they are characterising elements of mathematics. However, because practical needs do not require them anymore, one may argue that pupils should not be required to reach high computational skills: pupils should not become human calculators, but humans that are able to use calculators for mathematical tasks.

Within this perspective, the study presented here approached the educational problem of introducing pupils to algebra, starting from the consideration that available computer programs can easily execute algebraic calculations, thus, there is not a strong social need for pupils to acquire high algebraic computational skills. However, I believe that algebraic calculations are at the core of a great part of mathematical activities, and, I believe, if one has to efficiently use the computer for such activities, then he/she needs to be familiar with algebraic calculations. Again, pupils should not become human computers, but humans that are able to use the computer as a means for accomplishing algebraically, mathematical tasks. For this reason I believe that, even if computers execute algebraic calculations faster then humans, such calculations shouldn't be eliminated from educational practices, as they are instrumental to a correct and fruitful use of the computer as a mathematical instrument. However, because the evolution of technology suggests that computers will execute algebraic calculations faster and faster, better and better, I tried to individuate what are the aspects of algebra that, for what one can foresee nowadays, will not be changed drastically by technological evolution. The answer I gave to this question is that the theoretical aspects of algebra, won't be changed soon by the evolution of technology, in the sense that the idea of theory, and of proving within a theory, will keep being a founding element of mathematics. Thus I decided to explore the possibilities of introducing pupils to algebra as a theoretical system.

The process that led me to the individuation of the mathematical contents of my educational research, was deeply interwoven with my early research experience as educator. All started from the exploration of idea of using the computers for educational purposes, an idea which was, and is, quite popular in research in mathematical education. In my doctoral, and undergraduate, study, I had a chance to explore and appreciate many educational approaches based on the computer, or other technological devices. However, I felt immediately attracted by the idea of using computers as *instruments of semiotic mediation* as described within the vygotskian theoretical framework elaborated by Mariotti (**[51]**, Mariotti, 2002). Beside the fact that this was the approach I probably had a better chance to explore, what I really found interesting it was the idea that the computer doesn't teach, but that it is used by the teacher as a means for causing pupils' learning and guiding them toward meanings that are consistent with mathematics. It sounded to be like a sort of revenge of teachers: computers were not interpreted as substituted for the teachers, but as tools used by them. Moreover, the vygotskian framework focuses on the importance of language and semiotic systems in teaching learning processes highlighting how peculiar uses of signs can be used to influence such processes. Personally I have always been fascinated by the way humans communicate using languages and semiotic systems in general; thus I found myself naturally attracted by a framework in which semiotic activities are at the very core of the teaching/learning processes and are used, together with technological devices, as means for reaching educational goals. As a consequence, when I decided to begin the research I am presenting here, I was already strongly oriented toward a study of how to exploit computers, as instruments of semiotic mediation, to introduce pupils to algebra as a theory, within the vygotskian theoretical framework. In other words, my educational goals, and my research goals, stem out in parallel as tied one to the other:

- to introduce pupils to algebra as a theory.
- to study how it is possible to use the computer as an instrument of semiotic mediation in order to introduce pupils to algebra as a theory.
- To study how it is possible to use the computer as an instrument for semiotic mediation for mathematical educational purposes.

The research started by analysing a previous study, conducted by Mariotti (**[52]**, Mariotti, 2001), in which a geometry software, Cabri, is used as an instrument of semiotic mediation, to introduce pupils to geometry theory. From this analysis, the possibilities to set up a similar approach in the case of algebra theory, were explored. As a consequence there was a need to study what kinds of activities would have been meaningful for the educational goal, and what kind of computer software could be suitable for adapting to the case of algebra the educational approach presented by Mariotti. The preliminary study I conducted, resulted in the design and realization of an algebra software, L'Algebrista, and in the design of a sequence of educational activities to exploit it as an instrument of semiotic mediation to introduce pupils to algebra as a theory. The software, and the sequence of educational activities, where conceived in parallel, in this sense, L'Algebrista, is a software which was conceived explicitly for being used as an instrument of semiotic mediation to introduce pupils to algebra as a theory. Such a study resulted in the definition of an educational approach to algebra that up to now have been experimented for five years. Thanks to the feedback we got from the experiment, we have been able to re-elaborate the educational approach in itinere, and what we are going to present here is to be intended as a result of five years of research. In fact this research can be classified as *research for innovation*: theoretical and experimental studies maintain a dialectic relationship. A first set of research hypotheses frames the design of the teaching experiments, results, coming from those experiments, contribute to the evolution of previous hypotheses and consequently to the design of new experiments. Within such a process, a key role was played by the collaboration between researchers and teachers involved in the experimentation, which allowed a parallel evolution of the theoretical framework and of the class experiments.

This study is part of a larger project aiming at developing the theoretical construct of semiotic mediation and in this respect the outcomes of this study aim to contribute to this more general aim.

1.1. Summary

The first chapter is dedicated to the definition of the meanings attributed to some key words that will used in the rest of the dissertation. After such a preliminary part, literature will be reviewed, concerning research on computer and mathematics education. Drawing from this review some key ideas and problems characterizing this research will be stressed. In the fourth chapter we will restrict to the case of algebra, presenting a literature review on computers in algebra education, and describing some key aspects of algebra education in general. Drawing from such a discussion it will be presented our interpretation of algebra as a theory with a description of the key ideas of the assumed vygotskian framework.

In chapter 5 and 6 the software L'Algebrista will be presented, and it will be discussed in what sense it embeds the knowledge domain of algebra as a theory. In the following chapters it will be presented the design of the teaching experiment, describing both the sequence of proposed activities, and the educational strategies adopted together with the principles underlying them, in respect to the general hypotheses concerning semiotic mediation.

Finally, it will be showed evidence of the obtained results analysing data collected from the experimentation. The data will be analysed trying to exemplify and better formulate how L'Algebrista has been used as an instrument of semiotic mediation. The data analysed will show that certain educational goals have been reached, like that of introducing pupils to a theoretical perspective, and to symbolic manipulation as an activity of transforming expressions by means of the axioms of an algebra theory. Such objectives have been reached using L'Algebrista as an instrument of semiotic mediation, and the analysed data bring evidence of the potentialities of L'Algebrista for being used as such. We will show how the software has been effectively used by the teacher as an instrument for mediating meanings consistent to the educational goals. The semiotic mediation due to some particular elements of the software originated some particular meanings that evolved, under the guidance of the teacher, toward the aimed mathematical meanings. On the basis of the discussion of the experimental data, we will highlight the educational strategy of the experiment, and we will highlight some teacher's specific interventions to be considered as examples of use of L'Algebrista as an instrument of semiotic mediation.

2. Instruments and knowledge

This thesis concerning technology and education, implies a crucial use of words such as *object*, *artefact*, *tool*, *instrument*, *sign*, *and knowledge*. Some of them, in literature, are used with very specific, sometimes quite different, meanings, varying according to the framework considered; thus it seems necessary to start stating clearly the meanings we attribute to them throughout this thesis. The meanings of the words that we are considering here, may evolve throughout the development of this thesis, if that is the case, the new meanings will be defined. At this point we try to start with the most general meaning for each word, but paying attention to the constraints related to the aims of our work.

2.1. Object

The word object is a very generic one, which is described, in dictionaries, as having several different meanings, and we will consider the following ones in particular (see Appendix 9.4.5 for more details on the definitions found on some dictionaries):

Def 1: *Object as a physical or material thing*: in order to avoid confusion whenever we refer to this meaning we will use expressions such as "physical object" or "material object". In some cases, when the meaning is clear from the context, we may simply use the word "object".

Def 2: *Object as whatever can be the subject of intellectual or sensitive knowledge* (**[36]**, Garcia-Pelayo y Gross, pp. 393, 1994).

Def 3: *Object as fact, or idea considered as a separate entity*;

Def 4: *Object as focus of thoughts or action.*

Def 5: *Object as purpose*.

Observe that with these definitions we can consider the word *thing* as synonym of object in the senses Def 1:, Def 2:, Def 3:, Def 4:.

2.2. Artefact (or artefact)

Def 6: *An object made by human workmanship:* this could be either a material or non material, we consider *object* in the senses Def 1:, Def 2:, Def 3:.

An artefact, for us, will be an object which has been in some way produced by humans. As a consequence every artefact for us is an object, but not all the objects are artefacts; for instance, a stone, in general, is an object but not an artefact.

An artefact can be a physical thing such as a hammer, or a painting, but also any other product of human work. In particular, we will consider as artefacts objects such as computer programs, theorems, scientific theories, etc.

Finally we observe that we will assume that, in general, an artefact, being made by humans, is produced on purpose and the process that brings it to existence involves some human knowledge.

2.3. Instruments and tools

We will consider the words *tool* and *instrument* as synonyms (unless specified), both referring to the following definition.

Def 7: *a means whereby something is (or can be) achieved, performed, furthered, or done.*

Such definition is, on purpose, a very generic one, derived from an analysis of the definitions found in several English, Italian, and Spanish dictionaries (see appendix 9.4.5).

An instrument, for us, is a means used, on purpose, by an agent in order to achieve an objective, the agent (will usually be considered to be human). It can be anything, or any object[1], depending on what it is used for and how. Thus, we have either instruments that are artefacts, or instruments that are not artefacts; for instance, a hammer, or a stone, can be used as instruments to drive in nails, but the first one is an artefact, whilst the second is not, it is simply an object.

The nature of instruments, and the way they are part of human activities have been studied by researcher in the field of ergonomics, leading to several specific theories such as that of Rabardel (**[66]**, Rabardel, 1995) and those related to Activity Theory (see **[60]**, Nardi, 1995). Here we avoided explicit reference to such theories because, at this point of this thesis, we want a more generic definition to help us distinguish the main characteristics of the idea of instrument, those characteristics that differentiate it from other objects. In particular we refer to the following characteristics of instruments:

1. They are (or can be) used by an agent;
2. An instrument is (or can be) used in order to achieve, further, perform something, reach a target objective, etc.;
3. An instrument is a thing which can be identified as different from the agent[2] using it.

It is then possible to study either the nature of the object itself, the way it is used to achieve something, and the relationship between agent, instrument and how the latter is used.

2.3.1. The potential nature of instruments

When we specify that an instrument *can be* used, or that it is a means by which something *can be* achieved, we mean that an object doesn't need to be actually used in order to be considered an instrument. We consider as instruments those objects which we can figure out to use as instruments for any purpose. In other words, an instrument is an object we can think of as a means of reaching an objective, irrespective of whether we actually use it or not.

2.3.2. The relativity of instruments

The previous considerations lead us to specify that, from our viewpoint, the concept of instrument can be a relative one. In fact, it maybe the case that the same object is seen as an instrument by someone but not by other people.

We believe that the concept of instrument cannot be detached from a human, or a community of humans, who interpret it as an instrument to be used for to reach a goal. This, in particular, implies that different subjects, may see the same object as an instrument for reaching different goals, or the same agent can use a given object to achieve different things. For instance, one may use a ruler either to measure distances, or to trace lines, thus for different objectives. In this case (according to our definition) we may speak of different instruments, even if we refer to the same object.

Such relative characterisation can be of particular relevance when teaching and learning are concerned, because, in general, it may often happen that pupils and teachers using the same object, are using it to achieve totally different goals. The simplest example of this is when we are teaching someone how to use an object: for instance, when a guitar teacher plays the guitar to show one how

[1] An *object* in the senses of Def 1:, Def 2:, Def 3:.
[2] Here we mean whatever can be distinguished from the agent as entity, thus even a part of the agent, that is, a person can use his/her hand as an instrument for doing something. On the other hand, a person cannot use himself/herself, as a whole, as an instrument, while can be used as such by someone else.

to do it, he/she is not playing it with the objective of playing music, but with the objective of showing how to play; on the other hand, the pupil, may do exercises with his/her guitar in order to learn, not to play. Of course the objectives can somehow overlap, and both the pupil and the teacher may use the guitar as an instrument for producing music.

The last example introduces the idea of using instruments in teaching/learning processes, which will be central in this thesis. Thus we need to sketch a framework which takes into account either instruments, knowledge, and the relationship between them, highlighting how an instruments can "represent" and "evoke" mathematical knowledge. We already gave some characterizations of the idea of instrument, we will thus proceed in clarifying our view of knowledge, in order to be then able to study how instruments and knowledge can be related.

2.4. Knowledge

What do we mean by the word "knowledge"? This is too complex a question to be addressed in depth in this thesis, nevertheless, for our educational and research goals, we need to explore some characteristics of the concept of "knowledge". Therefore, due to the vastness of the subject, here we will address only the questions that we find relevant for our study.

The scenario that we are considering is that of school practice, thus its main actors are the pupils, and the teacher; moreover such a scenario is enriched by special artefacts (and/or instruments), introduced in the practice by the teacher for her/his educational aims, which we consider to be mathematical ones. In the following we are going to give our view of some aspects of *knowledge* in relation to all these elements.

2.4.1. Subjects and objects of knowledge

The first issue we address concerning knowledge come from the expressions "who knows what?" and “someone knows something”, that we take as primitives for our work, in the sense that we will not question the meaning of the verb “to know”, which we will consider as given.

Where school practice is concerned, there is a basic assumption which can be roughly synthesised by the following statement: "the teacher knows something that pupils do (or may) not know, and one objective of the practice is that when it is over, pupils will also know this *something*"[3]. Besides its triviality, this statement suggests the need of considering first of all the subjects, and the objects[4] of such knowledge (i.e. the *something* of the statement). We want to be able to say that ***what*** *a* ***pupil*** *knows* may be different from ***what*** *the* ***teacher*** *knows*, and we want to be able to individuate ***what*** it is that we want ***the pupil*** to know after education. Thus we start from considering the *who* and the *what* of our original question "who knows what?" (or equivalently of the *someone* and *something* of the expression "someone knows something"), and we take them as the basic elements allowing us to talk about knowledge; thus we ‘define’ them in the following way:

- **object of knowledge**: by this we mean the *what* or the *something* in the expressions mentioned above. Sometimes, we may refer to it as contents of knowledge, piece of knowledge, or simply knowledge;

- **subject of knowledge**: by this we mean the *who* or the *someone* mentioned above. In particular we are interested in human subjects of knowledge[5].

[3] How such transition, or change of state of the system class, happens is of course an open question, which we won't directly address, we will only study some possibilities to favour it using instruments.

[4] In the senses of Def 2:, Def 3:, Def 4:

[5] Even if it is not rare to hear statements with the verb *to know* where the subject is an animal, or a machine, we won't be interested in such cases.

For example, in the statement "the teacher knows mathematics", *the teacher* is the subject, and *mathematics* is the object of knowledge.

In short, we can speak of a *knowledge relationship* correlating a given subject A and a given object B, and expressed by the statement "A knows B". A subject in general knows many different things, thus is in relation with many objects of knowledge, we will refer to the whole set of such objects as the **knowledge of the subject**. For instance, when we talk about the knowledge of a person we refer to everything he/she knows[6]. Of course in this study we will be interested only in some subsets of a person's knowledge, for instance mathematics or knowledge about specific computer programs, or other subsets that we will define further on.

A subject of knowledge can be either a single individual, or a community of individuals, in the first case we will talk of **individual knowledge**, while in the latter case, we will talk about **social knowledge** (of the community), or **knowledge of the community** (similarly an object of knowledge can be individual or of a community). In a similar sense, Sutherland and Balacheff, talk about two types of knowledge, a socially shared and a personal one, which are tied in the sense that personal knowledge is interpreted as the individual's counterpart of the socially shared knowledge[7] (**[77]**, Sutherland et al., pp 2-3, 1999). Sutherland and Balacheff use the word *knowledge* for social knowledge, and the word *knowing* for personal knowledge. They define the first as an "intellectual construct, socially shared and institutionalised as efficient problem-solving tools" and define the latter as "personal intellectual construct related to knowledge" (ibid. 3), thus suggesting both a binding and a distinction between individual and social knowledge.

Similarly, we will interpret individual and social knowledge as distinct and separated, but related. The nature of the relationship between the social and the individual level has been studied according to a variety of views of knowledge, for instance in terms of distributed cognition, in terms of activity theory, in terms of intellectual interdependency. A deep analysis of this issue is beyond the scope of this thesis, here we limit ourselves to highlights what we assume to be the minimal conditions for the knowledge of a community to be named as such, as explained below.

A community is formed by persons, each with his/her individual knowledge; we assume that, for a given object of individual knowledge **A** (of a person), to be also an object of the knowledge of the community, the following conditions must hold:

4. all the members of the community know **A**;
1. each member or the community knows that the other members know **A** (equivalently, each member of the community knows that **A** is an object of the community knowledge).

If both the conditions hold, we will say that the object of knowledge **A** is *shared* among the members of the community, and we will say that it is part of the knowledge of the community.

This characterization is not to be interpreted too rigidly, where practice is concerned, in the sense that it may be the case that not all the members of a community know an object of knowledge, but it is nevertheless considered as an element of the social knowledge; for instance, not all humans know mathematics, but it is considered as an element of mankind's knowledge despite this.

Furthermore, because it may happen that the members of the community don't even know each other, a community may develop strategies, to make possible the sharing of knowledge, that do not depend on direct interaction. A simple example of this strategy is that of publishing news: when news is published in newspapers, it becomes known to a community, starting from the readers of

[6] Please observe that we are not making any assumption on how the knowledge of a subject is structured, organized, represented, acquired etc..

[7] Sutherland and Balacheff use the word *knowledge* for social knowledge, and the word *knowing* for personal knowledge. They define the first as an "intellectual construct, socially shared and institutionalised as efficient problem-solving tools" and define the latter as "personal intellectual construct related to knowledge" (ibid. 3), thus suggesting both a binding and a distinction between individual and social knowledge.

the newspaper, and ending up with other persons that may be informed directly by newspaper readers. Because the news is published on newspapers, the members of the community know that the other members of the community either know it, or can read it from a public source, the newspaper.

Later on we will see some of the strategies for knowledge sharing developed by the community of mathematicians, but before we need to say a few words on what we mean by learning, teaching and producing knowledge.

2.4.2. Learning, teaching and producing knowledge

Consider a situation where a *subject* firstly ***does not know*** a given *object* of knowledge, and if later the *subject reaches*[8] *a status* where *he/she* ***knows*** *it*, then we may talk about *learning*. We use the verb *to learn* in the generic sense of gaining knowledge (see **11.6.**), with no reference to any particular way to do this.

Similarly we will use the verb *to teach* in the generic sense of "cause to learn" knowledge (see **11.6.**) with no reference to any particular way to do this. Thus we may say, for instance, that a teacher teaches a pupil an object of knowledge when the first causes the latter to learn the given object.

A particular case of learning is when the subject is a community. According to our perspective (see 2.4.1), given a community **X,** and an object of knowledge **A**, where **A** to begin with is not an object of the knowledge of **X**. Suppose that in a second moment **X** gains **A** as an object of its community knowledge, then the two following conditions, at least, must hold:

5. all the members of the community learnt **A**;
2. each member or the community learnt that "the other members know **A**" (or equivalently each member of the community learnt that "**A** is an object of knowledge of the community").

If we consider in particular the second condition, it implies that the members of the community have been somehow interacting with each other, for instance by communicating, or by living a common experience, or by participating together in an activity etc. We assume that such interaction[9] is a prerequisite of **A** to become an object of the knowledge of the community **X**, in this sense we will say that the **A** (as object of the community knowledge) is a *product* of **X**. In other words, in the following, whenever we will talk of knowledge production, we will be referring to the learning of a community.

2.4.3. Mathematical knowledge

Where school is concerned, a key role of the teacher is to teach pupils a given set of objects of knowledge. Sutherland and Balacheff talk about *intentional knowledge* which they define as "the knowledge which the teacher desires to teach" (**[77]**, Sutherland et al., pp 4, 1999). The objects of such intentional knowledge are not just elements of the teacher's personal knowledge: they come from the knowledge of a community of persons who conventionally decided what pupils should learn in school. As highlighted by Sutherland and Balacheff, "schools have a commitment to ensure that pupils have access to knowledge which is needed by society", and "we have to accept the quasi-platonic nature of school mathematics" (ibid. pp. 3).

The community we consider for our study is that of Mathematicians, and we will refer to its community knowledge as *mathematical knowledge* or *Mathematics*; any object of intentional knowledge that we will consider, we assume to be also an object of such mathematical knowledge. The community of Mathematicians developed strategies for knowledge sharing based on the idea of

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[8] No matter how it happens.
[9] Whatever is its nature.

publishing (thanks to papers, books, computers etc.), but also criteria of acceptability of the published material. For instance, a published paper, to be accepted by the community for knowledge sharing, has to be written following certain rules, and using a specific mathematical language; furthermore, its contents have to pass a check of validity, based mainly on the ideas of mathematical theory[10] and proof. The criteria of acceptability of published material are themselves part of the mathematical knowledge, and function also as instruments to help single members to produce acceptable publishing material. They are specifically a characteristic of the mathematical community, and we cannot assume pupils to know them a priori.

Usually mathematicians cannot interact directly with each other, but they use means such as books, papers, computer programs, or other instruments. For instance, a mathematician can write a paper using a mathematical object of knowledge, the paper is then published, so that another mathematician can learn such an object of knowledge, which becomes shared by the two mathematicians; at this point, the paper can be used as a sign to represent the object of knowledge which, thanks to the paper, is shared by the mathematicians. Somehow we can say that the paper embeds mathematical knowledge, as we are going to explain in the following sections.

Suppose that an object of knowledge fulfils the above mentioned requirements for being accepted as a mathematical object of knowledge, and suppose that it is expressed in forms that are acceptable by the community of mathematicians, we will then consider such object of knowledge as mathematical, even if it is not published, even if it is not known to all the mathematicians.

If we consider an object of knowledge of a pupil, or of a class, then such an object, will never be published, but in case it is coherent with mathematical laws, and is expressed according to some mathematical language, then we will consider it to be a mathematical object of knowledge anyway, because, potentially, it could be shared with mathematicians. On the contrary, if this object of knowledge is at odds with mathematical rules, or is not expressed in a form which could be acceptable by mathematicians, then we will not call it mathematical.

Here we are being quite rigid on what we will accept as mathematical knowledge (in the sense of knowledge shared by mathematicians) in school practice, in the sense that we will not consider anything that doesn't obey the above mentioned conditions. Nevertheless we want to be able to talk of the mathematics of single pupils and of the mathematics of the class. By *class mathematical knowledge*, we mean the knowledge which is shared by the class and which the members of the class agree on considering as mathematical. Similarly, by *mathematical knowledge of a pupil*, we mean the knowledge that he/she considers to be mathematical. More in general, a subject's object of knowledge will be called a *mathematical object of the knowledge of the subject*, if it is considered as such by the subject. Of course, in the case of the community of mathematicians, then the mathematical knowledge of the subject corresponds to what we call Mathematics or *mathematical knowledge*, as defined above. Whenever we will speak of mathematical knowledge, without specifying the subject, then we will refer to the mathematical knowledge of the mathematicians.

For our educational perspective we assume as an aim of mathematical education, that of teaching the mathematics of mathematicians. In the following we will explore the possible relationships between artefacts and mathematical knowledge, in order to exploit them in school practice.

2.5. Embedded knowledge

Artefacts (Def 6:) are human products, and their creators employ some knowledge in making them. A painting, a guitar, a car, a cupboard, are all made according to some knowledge, as witnessed also by their historical evolutions which brought changes and advances thanks to human cultural and scientific development. In some sense, artefacts *embed* knowledge, be it painting techniques, or mechanics and engineering notions, or even simple notions such as that hitting a

[10] In the logical sense of the term

stone with another stone may cause its breaking into two or more parts. We may thus define *embedded knowledge* as follows:

Def 8: Given an artefact, by *embedded knowledge* we mean any knowledge that has been somehow employed by the creator of the object in the process that led to its existence.

This notion does not apply only to concrete artefacts or technical instruments, on the contrary, it applies to any artefact in general, could it be a hammer, or a more complex entity such a sign or a book. A special case of artefact is that of instruments, which are special artefacts conceived to be used for accomplishing given tasks. In this case, among the objects of its *embedded knowledge*, we will highlight those objects of knowledge which are strictly related to the correct functioning of the instrument to accomplish the task it was designed for. For instance if we consider a wooden abacus, we are not interested in the knowledge concerning how to carve wood, we will only be interested in the mathematical knowledge concerning the functioning of the abacus as instrument for computing. We may refer to this subset of the embedded knowledge of an instrument, as its *instrumental embedded knowledge*. Thus, if we are given a wooden abacus, and a plastic abacus, they have the same instrumental embedded knowledge.

An instrument, not only embeds knowledge, but it also can be a means, for its users, to access such knowledge. This makes instruments particularly interesting in education, in fact, if we consider an instrument incorporating a teacher's intentional knowledge, then we can suppose that pupil's, by using the instrument, may learn such knowledge.

We take as an assumption for our work the fact that that "a subject using an instrument may always learn something", which is witnessed by several theories describing how humans learn by using instruments. We introduce the concept of *evoked knowledge* defined as follows:

Def 9: Given an instrument[11], and a subject, by *evoked knowledge* (or evoked object of knowledge), we mean the knowledge (or object of knowledge) which is learnt by the subject by *using* the instrument [12] for accomplishing a given task.

Thus, an instrument, can be used to teach mathematical knowledge following the idea that a teacher can introduce the instrument into class practices requiring pupils to use it for accomplishing a task, and pupils will then learn some knowledge evoked by the instrument[13].

If our aim is that pupils learn, as evoked knowledge, the embedded knowledge of an instrument, we have to consider the fact that, given an artefact, pupils may interpret and use it as instrument in ways that do not coincide with the plans of the teacher and/or the creators of the artefact. For instance, the instrumental embedded knowledge of an artefact, concerns how to use the artefact as an instrument for accomplishing certain tasks; if a pupil uses it in different ways, or for accomplishing different tasks, we cannot aspect such instrumental embedded knowledge to be evoked.

Instruments have a relative nature (see 2.3.2), thus it is not even guaranteed that pupils see a given artefact as an *instrument* (at all) for accomplishing some tasks, and in case they do, it is not guaranteed that the tasks they use it for coincide with those foreseen and desired by the teacher.

When using an instrument for educational purposes, our main objective is that pupils learn some mathematical object of knowledge, so whatever the evoked knowledge is, we want it to be related to our mathematical intentional knowledge. This is not obvious: when we use a material instrument for solving mathematical problems, we often treat the instrument, and the object it acts on, as referring

[11] A similar definition can be given even for artefacts, and objects in general, but for our purposes we are interested mainly in instruments.
[12] Notice that according to this definition, when we talk of evoked knowledge, there is always a subject to which such knowledge is evoked by the considered object.
[13] Rabardell talks of internalization of schemes of use.

to mathematical instruments and objects. When a mathematician uses the commands of a symbolic manipulator to transform a screen expression into another one, he/she is interpreting the commands as mathematical transformation rules, and the expressions on the screen as mathematical expression; it is not a priori guaranteed that pupils do the same, they may interpret the expression on the screen merely as a writing on the screen, without any reference to any mathematical object.

It is like having two worlds, the material world of the instrument, and the mathematical world, and a semiotic link between them, in the same sense that mathematicians may interpret objects and actions in the material world as standing for objects[14] and actions in the mathematical one; an instrument can be thus interpreted as a sign standing for some object of mathematics. When a teacher asks a pupil to use a computer to study a function, he/she (the teacher) usually interprets the function on the screen, as a representation of a mathematical function; on the other hand, the pupil may interpret it merely as a an object of the screen. He/she may also interpret it as a "school function", which does not coincide with the mathematical object. But this is even another issue. A possible consequence of such different interpretations of the same object is that when the teacher thinks that learning outcomes are mathematical, in practice the pupil may gain only knowledge related to the computer program he/she is using. It is not guaranteed that pupils build, on their own, a semiotic relationship between the instrument and mathematics, as mathematicians do, almost spontaneously; to interpret an instrument as a sign referring to some aspects of mathematics requires some conditions to hold, as we will explore in the next section.

2.6. Characterisations of the concept of sign

On the subject of the characterisations of the concept of sign, usually addressed as *semiotics*, many studies have been carried out in the past, as described by Eco in his work (**[33]**, Eco, 1973; **[34]**, Eco, 1975) where he provides a global vision of semiotics. A complete account of the studies on the subject would be beyond the scope of this thesis, here we limit ourselves to acquiring some basic ideas from the theory described by Eco. The author, in his attempt to define a unifying and general theory of semiotics, describes various facets of the concept of sign; thus, for the aims of our thesis we highlight the following properties of signs.

2.6.1. Law of unlimited semiosis

Suppose for instance that we answer the question "what is a pencil?" by showing a pencil, then the object pencil becomes the signifying form for the same meaning expressed by the word "pencil" (ibid. 140). In other words, an object, such a pencil, which is not naturally conceived as a signifying form, can be used to represent something, and thus can be included in a semiotic activity in order to attain communication. This leads to the following principle expressed by Eco:

> *Any object to which a sign is referred, can become a signifying form for the meaning of the original signifying form. Thus objects that are merely signs do not exist, as any object can be taken as signifying form for another object[15] (ibid. pp 140).*

This law holds that whatever one wants to communicate , one can chose to represent it using any object; in particular, we may represent some aspects of mathematics, using some specific objects, for instance we can represent the idea of computing by means of a calculator, maybe just by showing it. This law suggests us the possibility of using an instrument, itself, as a sign of its embedded knowledge, or as a sign of a teacher's intentional knowledge.

But how can we create and use signs? What are the requirements for a sign to be effective in a communication process?

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[14] Here, of course, the word *object* is used a the wide sense, according to Def 1:, Def 2:, Def 3:

[15] Here we mean object general, thus in the senses Def 1:, Def 2:, Def 3:.

2.6.2. The relational, and conventional, nature of signs

Signs, according to Eco, are not physical entities, in fact they are correlations between a signifying forms and their meanings, as stated by the following principle:

> *Any sign correlates the field of expression (signifying field) and the field of content (field of meaning), both opposing, at their levels, substance and form. What differentiates signs is the articulation of their signifying form. Thus a sign doesn't exist as a physical, observable and stable entity, because it is the product of a set of relations. What we usually observe as a sign is just its signifying form (ibid. 142).*

This principle tells us that any sign, in order to be constituted, needs at least three elements: a signifying form, a meaning, and a relationship between them. For instance, in our previous example, we had a calculator (in the field of expression), the idea of computing (field of content), and a correlation telling us that the calculator represents computing. Here we may argue that it is not obvious that by showing a calculator to a person, will cause he/she to interpret it as (understand that we are talking of) computing in general, maybe he/she won't even understand that we want to communicate something. When we interpret an object as a signifying form, for a given meaning, it is not obvious that our interlocutors share our interpretation, and in case they don't, then we may fail in communicating with them. So how can we be sure that we share the meanings of the expression that we are using when we are trying to communicate with other subjects? A possible answer can be found in the principle of conventionality of signs described by Eco (ibid. 142):

> *In a sign, the signifying form is associated with its meaning by a conventional decision, thus following a code.*

Here *conventional* does not mean *arbitrary*. There could be good reasons if a specific signifying form is considered suitable to represent a specific meaning, but there must be some kind of agreement between the subjects involved in the communication. Take for instance the example of children making drawings, they may use signs that we do not share, and we have a chance of understanding the drawings only after having been explained what they intend to represent. So, going back to our example of the calculator, we now may suppose that if I show a calculator to the reader of this thesis, then he/she may understand that I am referring to the idea of computing, because I previously explained what I mean when I show such an object. In other words, according to our common experience, the reader and I somehow share a code where calculators are associated to the idea of computing.

In conclusion, codes give shared rules for matching the elements of the field of expression with the elements of the field of contents. Such rules for matching are indispensable for the existence of a code, and must be conventional and socialised[16]. According to Eco, codes are necessary and sufficient for the existence and consistency of a sign:

A sign is defined by its possibility to institute the relation signifying-meaning on the basis of a code; a sign is not defined by the fact that its signifying form has been outputted intentionally (ibid. 143).

As a consequence, for pupils to use a semiotic relationship between two entities (for instance an instrument and mathematics, see 2.5.), they must know the code it belongs to; if it is only the teacher who knows this code, then we cannot expect pupils to spontaneously build the desired semiotic connection.

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[16] Here it doesn't matter if the rules are the result of an imposition or not, what it is important is their presence and that they are known (maybe implicitly) to the subjects involved in the communication, when the communication occurs.

2.7. Educational and research implications

The immediate consequence of the perspective we described is that when we plan to use instruments in education we have to take careful account of:

- Our intentional mathematical knowledge.
- The embedded knowledge of the instruments.
- What knowledge can be evoked by the instruments.
- The relationship between the knowledge embedded in the instruments and our intentional mathematical knowledge: *is there any coherence? Any Contrast? Any Relationship at all?*
- The relationship between the knowledge that can be evoked by the instruments and our intentional mathematical knowledge: *is there any coherence? Any Contrast? Any Relationship at all?*
- How pupils interpret the given artefact: do they use it as an instrument in the ways, and for the tasks, intended by the teacher?
- What connection pupils see between the knowledge evoked by the instruments and mathematics.

In summary, the key idea of this perspective is that when using instruments some knowledge can always be learnt, but what it is not obvious is what kind of knowledge this is. We can plan the use of objects for educational purposes, but we have somehow to keep control of the produced knowledge. In particular, in the case of mathematical education, due to its conventional nature, we should always question the relationship between such produced knowledge and mathematics itself.

The perspective that we introduced in this chapter, will be refined in the following chapters on the basis of a literature review on how instruments have been, and are, used in mathematical education.

3. Computers in mathematics education

In this chapter we are going to give an overview of some aspects of the evolution of the use of computers in mathematics education, taking a research oriented perspective. We are going to analyse such evolution trough different lenses, starting from design metaphors, passing through language and manipulation matters, and ending up with a pedagogical perspective.

The aim of the chapter is not only to describe and refer to past research, but it is also to define the research space where this work is going to be situated. We are going to individuate both problems to be addressed and key ideas to be pursued. Thus, after reviewing the literature, we will comment on it through the lenses of our educational goals.

3.1. The lenses of metaphors

Along the history of the use of technology in education we find a line of evolution on the bases of the different metaphors used to describe/design the relationships, and interactions, between the human, the computer, and knowledge. Such perspective is relevant because highlights the positions of the different theoretical frameworks respect to knowledge, pupils, teachers, community culture and the relationships among them. Comprehensive descriptions of such evolution of educational approaches are given by Bottino (**[11]**, Bottino, 2001, pp. 13), and Bottino and Chiappini (**[12]** Bottino and Chiappini, 2002, pp. 758), drawing from their work we will point out those aspects that we consider to be relevant to for our study. Three main orientations, can be considered, according to three different metaphors: the *transmission metaphor*, the *learner-centred metaphor*, the *participation metaphor*.

3.1.1. The transmission metaphor

The *transmission metaphor* is based on the idea that knowledge can be transferred from one person to another, and when technology is concerned, from a person to an object, and from an object to a person. The cultural context is that of behaviourism which, in fact, influenced the first ways in which the computer had been used for educational purposes. Learning was seen as the "induction of a required behaviour according to the well-known model 'stimulus response'" (**[11]** Bottino, 2001, pp. 13). The reference to such model led to the design of systems such as those usually referred as to *drill and practice programs* and *tutoring systems*.

Drill and practice programs consist mainly of automated ways to submit exercises to pupils, users are faced with questions to answer, and usually get feedback on the correctness of the answers. As Bottino observes, "they usually employ some form of questioning strategy and often use some gaming techniques for encouraging participation and motivation", and "ordinarily […] are not used during the normal class activity but for individual training 'ad hoc' hours at home" (ibid., pp. 13).

Tutoring systems, differently from drill and practice programs, are often base on an information processing approach to learning. Their design ascribes importance to reinforcing memorisation, presenting objectives, specifying prerequisites, eliciting and assessing performance. Given a topic, they include related content instruction, and present questions that, to be answered, require the user to employ concepts or rules covered in the instructional sequences. The given feedback is mainly diagnostic, aiming at identifying processing errors and prompting remediation or recasting of the instructions. Such systems are conceived as "'stand alone' systems, designed as a single learner's private tutor" and "their use in classroom practice is limited since they are often perceived more as replacements of teachers than as tools to help them in their work (ibid., pp. 13).

According to Bottino both kinds of computer programs revealed to produce quite limited advantages; in fact they do not substantially change the way their users interact with a given object of knowledge, and do not contribute to furnish a learner with new ways to give meanings to the related concepts. The system is conceived as an "environment where knowledge is transmitted in order to be acquired by the user" (ibid., pp. 13-14).

Paradoxically, despite, and because of, its limited educational advantages, the transmission metaphor played a key role in the evolution of educational computing research, as Bottino and Chiappini observe:

> "One of the major forces driving change has been the assumption that meanings are lost if learning is simply the transmission of information".
>
> ([**12**] Bottino and Chiappini, 2002, pp. 758).

3.1.2. User centred metaphor

Within the *transmission metaphor* paradigm the user of a given educational technological system is mainly a receptor, while the object itself is in charge to transmit knowledge. Thus, within this paradigm, the study of the system itself has great relevance because it has, to some extent, to contain some knowledge and be able to transmit it. Such an unbalance of focus was reversed when the interest on constructivist theories increased, leading to a shift of attention from the system to the user, to the internal aspects of the learner ([**11**], Bottino, 2001, pp. 14).

Many authors use expressions such as "learner-centred systems" and "problem-based learning", and, in general, view learning as based on active exploration. The learner has to be in some way immersed in the topic and be involved in problem solving activities relevant to the topic. Such involvement is supposed to motivate the learner in seeking new knowledge and acquire new abilities (ibid., pp. 14; [**12**] Bottino and Chiappini, 2002, pp. 758).

Given a topic, one may think of creating an environment whose objects have some relationship with the topic and where learning may occur by exploring the environment. Such an idea is at the core of the concept of microworld, introduced by Papert, an environment that is built around a given domain which has to be explored by interacting with the program. A detailed history of the concept of microworlds can be found in Noss & Hoyles ([**25**] R. Noss and C. Hoyles, 1996, pp. 63-67).

Where a microworld is concerned, a crucial role is assumed by the objects that are made available to use through the interface of the microworld:

> "Papert defined them as a transitional computational objects, that is objects which are inbetween the concrete and directly manipulated, and the symbolic and the abstract"
>
> ([**11**], Bottino, 2001, pp. 15).

Thus, for educational purposes, it is important to consider the epistemology of the *transitional objects* in order to evaluate microworlds and "distinguish between potentially powerful environments and environments less appropriate for exploration" (ibid., pp. 15).

However, if on one hand, epistemology played a crucial role in the design and choice of microworlds; on the other hand, when learning situations and educational research were concerned, great attention was given to learners behaviours. The objective was to design and analyse learning situations which could favour the emerging of knowledge from the interaction between the student and the computer environment.

To sum up, the focus was both on the technological systems and on the learners, their roles were different, but both crucial for design and implementation of educational activities. Within such a framework significant results have been produced by experimentation, but its impact on school practices was weaker then expected, mainly because classroom practices had not been adapted in order to exploit the new technological tools:

> "high expectations regarding ICT-based tools potential to drive change and innovation at school remain largely unfulfilled. One of the main reasons for this […] is that technology has often been introduced as an addition on to an existing, unchanged classroom setting"
>
> ([**1**] Bottino, 2001, pp. 15).

3.1.3. Participation metaphor

The previously described paradigms focused mainly on the couple technology-user couple, and the relationship, and interactions, between them. This turned out to be limiting, for the purposes of education, in fact, technology itself turned out to not to have the power of giving greater meaning to educational activities. Research showed a need to extend the focus, when a tool is concerned, its pedagogical significance cannot be defined by taking into consideration only its characteristics, but rather by considering aspects that are external to the tool itself ([**12**] Bottino and Chiappini, 2002, pp. 758-759).

There is then a need to develop, together with new technology, specific educational paradigms aiming at exploiting at best the new resources, in order to improve teaching and learning activities. Technological tools were often used on the assumption that their use would lead to educational improvements simply because the tools themselves were considered to be "good". Such a simplistic approach led to initial high expectations, followed by disillusionment. It is a shared opinion that That happened because there was a lack of understanding of the conditions under which educational tools, and their use, might be meaningful.

This issue, in recent years, has represented a major topic in the debate conducted by researchers in the domain of educational computing. The ongoing discussion shifted the focus from cognitive theories to other perspectives, less focussed on the individual, and more oriented to highlight the social nature of cognition and meaning production (ex. Activity theory, Situated Action Models and Distributed Cognition; see [**35**] Nardi 1996). Within this theoretical frameworks, practice is viewed as interlaced with learning, and meaning is interpreted as interlaced with the practices and the contexts in which it is negotiated ([**1**] Bottino, 2001, pp. 16).

The new trend influenced also the design and use of technology which was no longer conceived merely as a means for the development of specific abilities and/or the accomplishment of particular tasks. The whole teaching and learning activity was being taken into account, included the long term processes that are needed to develop complex articulated knowledge and that can hardly be analysed considering only the student-software unit. The idea is that of interpreting learning not only as an individual construction developed during the interaction with the computer, but also as a social construction developed within the whole learning environment .

To sum up, we assisted to an evolution of the idea of *learning environment*, which led to the inclusion of the whole learning situation, starting from a situation where the tools itself was considered as the whole learning environment. This focus on the whole teaching/learning context leads to reconsider the role of the teacher, and the attention is put not only on software design, but also on how it can be used for specific educational purposes:

> "Consequently there is an increasing interest in aspects related not only to software design but also in the definition of ways of use suitable for exploiting software features in order to accomplish meaningful teaching and learning activities"
>
> ([**1**] Bottino, 2001, pp. 17).

3.1.4. Résumé of significant ideas and problems

We outlined the evolution of the use of technology for educational purposes; such description highlighted both key ideas and problems concerning the considered approaches. If on one hand, some of the key ideas, turned out to be powerful and have not been discarded by the evolution of the subject. On the other hand, some of the problems related to some approaches pushed researchers toward new directions.

We observed an evolution of the literature from the point of view of the unit of analysis considered, and of the roles played by the technological systems and by its users. Researchers started from considering only the couple user-system, and ended in enlarging the unit to the whole learning context within the participation metaphor approaches. At first the main actor was the system, then, under the influence of constructivism, this role was played by the user, and the design and use of learning environments was conceived to adapt to the user and serve the user development in some way. Finally, in recent years, as the whole learning situation has began to be considered, we witnesses the inclusion of other elements which may play roles as important as those played by systems and users; for instance we witnessed an increasing importance to the role played by the teacher. Thus, because our focus is on mathematics education, and on the use of specially designed computer software, taking into account of the discussion in chapter 2. , and its educational implications (see 2.7.), we will:

- *Take the whole learning context (including the teacher) as the principal unit of analysis*, thus referring to the participation metaphor paradigm. This is not only because of the pedagogical and cognitive aspects previously highlighted, but also because of the social nature of mathematics (see 2.4.3), thus of our educational goals. In particular, with the term learning environment, we will refer to the whole learning situation.

An issue that will be central for our study is that of pupils' learning of teachers' intentional knowledge (2.4.2). Within the *transmission metaphor* knowledge was assumed to be acquired (thus learnt) by pupils simply thanks to transmission of contents, while within the *learner-centred* approach, knowledge is assumed to be recreated/reconstructed (thus learnt) by pupils by working within microworlds or in general within specially designed learning environments. The latter is a constructivist principle that proved its validity, but that showed also some weakness. A crucial point, here, is the coherence of the knowledge built by pupils with the knowledge the teacher is trying to teach (intentional knowledge). If only the system user-microworld is considered, then such coherence can be ascribed only to the user and to the nature of the microworld and the interactions with it. Accurate epistemological studies of the system (even during the design phase), may point out some kind of potential knowledge which in some way is embedded (see 2.5.) in the microworld. Nevertheless such knowledge may not necessarily be learnt by the user: it may happen that users do not relate at all, the activity within the microworld with what they are supposed to learn, as showed, for instance, by Guin and Trouche (**[35]**, Guin and Trouche, 1999).

Finally, the *participation metaphor* approach considers knowledge learning as a result of social practices, which, actually, can also include working within microworlds.

From the above discussion, coherently with the concluding remarks of chapter 2. (see 2.7.), we take, as candidates for our theoretical framework:

- The principle that working with microworld leads to a learning outcome that can be significant in school practice.

- The principle that learning may happen through social practices.

Given such principles one key question that we will try to address is:

How is it possible, when working with microworlds, to produce learning outcomes that are coherent to a given educational goal (including a mathematical one)?

This question will be partly addressed in the following literature review, and an attempt of answer will be given later on, in terms of the description of our framework of reference.

3.2. Programming environments and microworlds

Microworlds, either in the form of computer programming and in the form of direct manipulation environment, played an important role in the history of mathematics education. Noss and Hoyles, in their book "Windows on mathematical meaning" (**[61]** Noss and Hoyles, 1996) trace a history of the evolution of the idea of computer programming within mathematics education, focusing in particular on the idea of microworld. Within this section we will draw, from their work.

Noss and Hoyles distinguish between two main kinds of software: on one side, software designed to deliver existing mathematical curricular presenting knowledge in a wrapped form acceptable for educational consumption; on the other side there are computer applications pointing toward a redefinition of school mathematics in more learnable forms (**[61]** Noss and Hoyles, 1996, pp. 54).

The first kind of software, in general, resides within the transmission metaphor (see 3.1.1) and according to Noss and Hoyles is focused on the word "teach" resulting to be of little pedagogical interest, as it doesn't offer more then a human tutor. Moreover, this kind of software fails in providing learners with new means of expressing their mathematical ideas, thus, using the author's words, it "also fails to open any windows onto the process of mathematical learning" (ibid., 54).

The second kind of software, on the contrary, seems to have more interesting pedagogical potentialities. The attention is put on the idea of *articulation* as a process that allows pupils to "create" (thus learn) mathematics and that reveals this "act of creation" to observers (ibid. pp. 54). In other words, technological systems can be interpreted as potential means for articulation enhancing pupils' expressive power. Within this class of software, a historically important example is that of the LOGO programming environments.

3.2.1. Programming vs. direct manipulation: toward new expressive tools

According to Noss and Hoyles there is a long pedigree to the idea that computer programs provide learners with a vast canvas where to sketch half-understood ideas, and where to assemble semi-concrete images of the mathematical structures he/she is learning. Early research showed how a promising line of enquiry, that of the expression of mathematical ideas in the form of computer programs, could be. Nevertheless we assisted to a relative isolation of the programming community within mathematics education, mainly due to the notion of programming held within the wider educational culture: a programming language is in general viewed simply as a mean for writing programs, lines of text are entered and output is obtained. This makes it appear arcane and complex if compared to multimedia systems or direct manipulation interfaces (ibid., pp. 55).

However, developments of the nature of programming have changed the situation, and the notion of programming shifted to include visual and iconic rather then purely textual means of combining elements; in other words, there has been a tendency to furnish programming environment with the attractive features of other popular computer systems. The authors cite the example of system called *Boxer,* that "aims to provide the user with much greater expressive power", as explained by DiSessa (**[25]**, diSessa 1998):

> "We are trying to produce a prototype of a system that extends with computational capabilities the role now played in our culture by written text. It should be a system that is used by very many people in all sorts of different ways, from the equivalent of notes in the

margin, doodles and grocery lists – all the way to novels and productions that show the special genius of the author, or the concerted effort of a large and well-endowed group. In a nutshell, we wish to change the common infrastructure of knowledge presentation, manipulation and development (diSessa 1998b, p.3)."

([**61**] Noss and Hoyles, 1996, pp. 55-56).

On the scale of the debate concerning the advantages and disadvantages of direct manipulation versus text-based interfaces, we find on one side the friendliness of interface, while on the other side we find expressiveness of programming languages. Direct manipulation may foster a greater sense of engagement of the learner with the screen objects, whilst programming may better enhance pupils' descriptive language and communication (ibid., pp. 56).

The antithesis between programmability and direct manipulation has been broken down by the development of systems which tend to exploit the strengths of both. The new situation has been described by Eisemberg ([**35]**, Eisemberg, 1995) who talks about *programmable applications* as integrating the key features of both, programming and direct manipulation:

"Programmable applications … are software systems that integrate the best features of two important paradigms of software design – namely, direct manipulation interfaces and interactive programming environments. The former paradigm – popularly associated with menus, palettes, icon-based interaction techniques and so forth – stresses values of learnability, explorability, and aesthetic appeal; the latter, by providing a rich linguistic medium in which users can develop their own domain-oriented 'vocabularies', stresses values of extensibility and expressive range. (Eisemberg, 1995, pp. 181)"

(as cited in **[61]** Noss and Hoyles, 1996, pp. 56).

The central idea, is that of opening up to the learner the expressive power of programming as a means to navigate and reconstruct a knowledge domain, as a "tool for expression and articulation". However, even in the case of a system with a friendly interface, we can't avoid questioning how programming environments can favour learning and mathematical expression:

"how crucial is this for learning mathematics? Most importantly of all, how far can focus on programming in particular serve as a window on mathematical expression in general?".

(ibid, pp. 57).

3.2.2. Computers and conviviality

Noss and Hoyles focus on the vision of programming as a tool for expressing and articulating ideas, and, drawing from Illich's work, introduce the idea of *conviviality* and *convivial tools*:

"Illich has called such tools *convivial*: conviviality is a question of *meaning*.

> To the degree that he masters his tools, he can invest the world with his meaning; to the degree that he is mastered by his tools, the shape of the tool determines his own self-image. Convivial tools are those which give each person who uses them the greatest opportunity to enrich the environment with the fruits of his or her vision.
>
> (Illich, 1973, p.21)

The extent to which a tool may be seen as convivial is the extent to which the use of the tool creates meanings for its user, catalyses intellectual experience and growth. […] it is our view that the computer is a tool which can be used to enrich the social psychological space of the individual with the fruits of his of her (mathematical) vision"

([**61]** Noss and Hoyles, 1996, pp. 57).

The authors see a natural way to link conviviality with programming, they cite again Illich:

> "Illich says:
>
> > Tools foster conviviality to the extent to which they can be used, by anybody, as often or as seldom as desired, for the accomplishment of a purpose chosen by the user. …*They allow the user to express his meaning in action.*
> >
> > (Illich, 1973, pp. 22-23, emphasis added)
>
> Meaning expressed in action. Here is the heart of the matter […]"
>
> (**[61]** Noss and Hoyles, 1996, pp. 57).

If we interpret computers as tools, then we may interpret them in terms of conviviality, and use them to express meaning in action; furthermore, according to Noss and Hoyles, the computer is a special tool because of its relationships with mathematical formalism:

> "the computer is a special tool in which action involves the formal use of language, and where the usual polarities – meaning and precision, intuition and formalism, conviviality and rigour – do not hold. For mathematical learning this is a crucial facet of the computer's role. As Papert points out:
>
> > Most children never see the point of the formal use of language. They certainly never have the experience of making their own formalism adapted to a particular task. Yet anyone who works with computers does this all the time.
> >
> > (Papert, 1975, pp. 220)"
>
> (**[61]** Noss and Hoyles, 1996, pp. 58-59).

Thus computers can be interpreted as tools, and acting with them they can be thought as enriching users' expressive power, in a way that is mathematically relevant because of its relationships with formal languages.

The authors conclude the section observing firstly that tools are cultural objects, which are not passive and play an active role in the culture they are inserted in. Secondly they observe that the level of conviviality of a tool doesn't depend strictly on its characteristics, but depends on the relationships with its user:

> "First, tools are cultural objects. Tools are not passive, they are active elements of the culture into which they are inserted. Second, the extent to which a tool is convivial is determined by the relationship of the user to the tool, not by any ontological characteristic of the tool itself."
>
> (**[61]** Noss and Hoyles, 1996, pp. 58-59).

The availability of a tool is not enough to enrich the user's expressive power, in order to do that, a tool has to enter the users' thought's, actions and language:

> "For a tool to enter into a relationship with its user, it must afford the user expressive power: the user must be capable of expressing thoughts and feelings with it. It is not enough for the tool to merely 'be there' […], it must enter into user's thoughts, actions and language. Expressive power opens windows for the learner, it affords a way to construct meanings"
>
> (**[61]** Noss and Hoyles, 1996, pp. 58-59).

3.2.3. Why programming?

Noss and Hoyles sum up their educational and research reasons for focusing on programming, pointing to the fact that formalism, and rigour of the computer, together with the fact that it offers the user a language to talk about mathematics.

Programming computers constrain users to a certain level of formalism which is somehow comparable to that of mathematics, thus educationally significant. In fact, the compactness and rigour of expression involved in programming is not isomorphic, but is compatible, to that demanded by official mathematics. It is precisely such difference, between mathematics and programming, that may open up mathematics to a variety of learning styles and expression:

> "There is a degree of compact and rigorous expression involved in programming which if not isomorphic to, is at least comparable with that demanded by official mathematics. Of course, it would not do correlate mathematics and programming merely on the basis of similarities between notation systems. On the contrary, it is partly the *difference* between the two which give programming the opportunity to open up mathematics to diversity of learning styles and expression."
>
> (ibid., pp. 62).

Finally, Noss and Hoyles observe that, due to its peculiar ways of expressing mathematical relationships, programming, as opposed to inert representations, may open strategic apertures for children:

> "If we pay careful attention to the design of Logo-based worlds, we might unlock strategies for children which are simply closed in conventional media. That is, by throwing into relief particular ways of expressing mathematical relationships, the programming environment may open strategic apertures for children – ways of expressing which are available with the computer and closed in inert representations."
>
> (**[61]** Noss and Hoyles, 1996, pp. 62).

3.2.4. Microworlds

According to Bottino (**[11]** Bottino, 2001, pp. 15) , the meaning of the word *microworld* is not a standard among researchers, but there is a certain level of agreement on what characterises such kind of systems. Here we chose a characterisation, given by Balacheff and Kaputt (Balacheff & Kaput, 1996, p. 471), which is generic enough to fit also Noss and Hoyles' discussion:

> "A microworld consists of the following interrelated essential features:
>
> i) a set of primitive objects, elementary operations on these objects, and rules expressing the ways the operations can be performed and associated - which is the usual structure of a *formal system* in the mathematical sense.
>
> ii) a *domain of phenomenology* that relates objects and actions on the underlying objects to phenomena at the 'surface of the screen'. This domain of phenomenology determines the type of feedback the microworld produces as a consequence of user actions and decisions (Balacheff & Sutherland, 1994)"
>
> (**[1]**, Balacheff & Kaput, 1996, p. 471).

Noss and Hoyles present a genesis of the idea reporting how the meaning of the word microworld evolved during a couple of decades (for more details see **[61]** Noss and Hoyles, 1996, pp. 63-67). The key idea of the concept of *microworld* is that of an environment, characterised by a knowledge domain which thus becomes explorable .

At the core of the relationship between the user and the *knowledge domain* there are the objects of the interface that are available to the user. Papert termed them *transitional objects* "standing between the concrete/manipulable and the formal/abstract" (as cited by Noss and Hoyles, ibid., pp. 64). Such objects are the means of the interaction between the user and the environment, thus, between the learner and the knowledge domain .

A microworld is thus an environment where exploration is possible thanks to *transitional objects*, but where such exploration is constrained in ways designed to promote learning; knowledge is assumed to be reflected in the system, in its elements, relationships and structures, and it is assumed to be evolving by means of actions within the system. In particular, a microworld has to be able to reflect evolvable knowledge along the course of activities, thus it has to be evolvable itself:

> "Exploration is necessarily constrained but in ways designed to promote learning; knowledge is not simplified, it is recognised as complex, interrelated and evolving in action. These facets are reflected in the structures of the system, particularly in its extensibility – the extent to which the elements of the microworld can be combined, recombined and extended to form new elements"
>
> (**[61]** Noss and Hoyles, 1996, pp. 65).

To sum up three main characteristics of microworlds are: that they embed a knowledge domain model; that they offer transitional objects to the user to act with; that they can be extended[17].

In the effort to study the possible impact of microworlds on the supposed gap between learner and a given knowledge domain Noss and Hoyles (ibid., pp. 67). consider the two following requirements:

- *Concreteness*: "we must try to construct situations which are sufficiently concrete" where by concrete they mean "well connected to what the learner already knows"
- The idea of *reconstructability*: "if we design structures that allow the students to (re)construct new objects and relationships out of old ones, we can increase the likelihood that will achieve some visible and tangible representation of the state of the student's thinking and be more able to observe mathematical thinking-in-change"

They focus on the nature of computational objects as the central elements in a microworld, the choice of which is critical:

> "The choice of such objects and the ways in which relationships between them are represented within the medium, are critical. Each object is a conceptual building block instantiated on the screen, which students may construct and reconstruct."
>
> (**[61]** Noss and Hoyles, 1996, pp. 68).

[17] Other characterisations of the term microworld are available, but they may be too restrictive for our purposes, like for instance that given by Bottino, who includes also the concept of direct manipulation: "For example, microworlds should provide the user with a number of primitives […] that can be combined in order to produce a desired effect. They should embody an abstract domain described in a model, and offer a variety of ways to achieve a goal. Moreover they should allow the direct manipulation of objects, etc" (**[1]** Bottino, 2001, pp. 15). Such characterisation may be at odd with Noss and Hoyles' discussion which is focused on programming. For the moment we keep a more general characterisation of the concept, leaving out "direct manipulation" aspects.

The effectiveness of computational objects depends on their capability to stimulate learners and on their capability to intuitions, understandings and personal images, together with preferences and pleasure. By means of computational objects, microworlds can be used to fill the gap from users' existing experiences and static formal systems, a gap which is often too great for learners to engage directly with such systems. On the other hand, the direct interaction with the computational objects offers a chance of connecting learners' knowledge with the knowledge domain represented by the microworld . In particular, mathematical microworlds should forge links with mathematical objects and relationships, this aspect distinguishes them from a simply playful exploratory world which is mathematically uninteresting .

How such links are built is an opened question, but language seems to be a good candidate to link microworlds and mathematics:

> "[…] a computational world can be *autoexpressive* – it can contain the elements of a language to talk about itself. The definition of what such a language might be like is broad and broadening all the time: but we are not prepared – at least for the time being – to let go of it altogether: at root, it is the language, the program, which allows the most obvious link between computational and mathematical discourses"
>
> ([**61**] Noss and Hoyles, 1996, pp. 68).

3.2.5. Some peculiar pedagogical contributions of microworlds

current literature provides a wide account on the possible pedagogical contribution of microworld. A rich and comprehensive review can be found in ([**62**] Pratt, 1998, pp. 57-61), where Pratt describes some key pedagogical contributions of research concerning microworlds in this section we will report on some of such contributions that we find relevant for our work.

*3.2.5.1.**Quasi-Concrete Objects***

Within microworlds, the objects that are visible on the screen, are in a sense twofold. On one hand, like mathematical constructs, they may be defined according to formal rules; on the other hand they are visible, and to some extent manipulative, like tangible objects. According to Turkle and Papert, the computer has the power to make the abstract concrete, bridging formal systems and physical things, because of its double faced nature. Its objects obey and are defined according to formal rules, but can be perceived as physical, allowing a sense of physical manipulation, that in the case of mathematical objects is usually felt only by expert mathematicians:

> "The computer stands betwixt and between the world of formal systems and physical things; it has the ability to make the abstract concrete. In the simplest case, an object moving on a computer screen might be defined by the most formal of rules and so be like a construct in pure mathematics; but at the same time it is visible, almost tangible, and allows a sense of direct manipulation that only the encultured mathematician can feel in traditional formal systems"
>
> ([**81**] Turkle & Papert, 1991, pp. 162).

Mathematical objects are represented on the screen, they keep some of their formal aspects and mathematical behaviours, but at the same time it is possible, even for an uncultured learner, to treat them as if they were concrete objects:

> "The image of the quasi-concrete object opens up the possibility of children manipulating and using mathematical objects, which conventionally would be seen as abstract but which are represented as if concrete on the screen. "
>
> ([**62**] Pratt, 1998, pp. 58)

Once mathematical objects are represented as if concrete, it is then possible to interact with them, thus to develop a direct relationship with them, a step which is considered fundamental for meaning construction.

> "[…] as children interact with quasi-concrete objects, they develop a relationship with the object. The familiarisation of the object is a fundamental part of the construction of meaning."
> (ibid., pp. 58)

*3.2.5.2.**Using Before Knowing***

Mathematical objects can be interpreted as tools for solving problems or to be used for accomplishing some task. According to Pratt, the natural way to learn about tools, in general, is to use them for particular purposes, a practice leading to learn "the effectiveness of the tool, its limitations, how well it serves that purpose and sometimes we may gain some understanding of how it works". According to the author this is not the case for mathematics in school where the situation is reversed: the learner is firstly faced with the problem of generating a mathematical object, and only after that phase he/she is faced with activities where the object is used as a tool for accomplishing some tasks:

> "Mathematics uniquely has always been represented as different. In schools, mathematics is a subject where you learn how to generate the object before you use it. In practice, more often than not, the former task proves too difficult, especially when disconnected from purpose, and so we never reach the second stage of using the piece of mathematics."
> (**[62]** Pratt, 1998, pp. 59)

The computer offers a way to reverse the situation, so that, *using* may precede *generation*, as Pratt observe citing Papert:

> "The computer offers the possibility of turning the learning of mathematics round so that using precedes generation, thus bringing mathematics more into line with natural ways of learning. Papert has recently referred to these ideas as the *Power Principle* (Papert, 1996).
>
>> The principle is called the power principle or "what comes first, using it or 'getting it'?" The natural mode of acquiring most knowledge is through use leading to progressively deepening understanding. Only in school .... is this order systematically inverted. The power principle re-inverts the inversion. (p. 98)"
>
> (**[64]** Pratt, 1998, pp. 59)

from these arguments we can abstract The following principle: if we interpret mathematical objects as tools, then it is possible to learn about them by their usage, and the computer systems / environments offer a chance to set up activities where mathematical objects become usable.

*3.2.5.3.**Integrating the Informal and the Formal***

According to diSessa, Pratt stress the need of building connections between formalisations and use of objects, aiming at fusing them:

> "[…] diSessa has suggested that we incorporate versions of the formal representations of the mathematical objects in such a way that the child may be able to make connections between the various formalisations and their informal use (diSessa, 1988).
>
>> Building analytic or other formal tools right into experiential environments should become more and more a standard part of microworld design.....The idea is not to juxtapose experiential and formal points of view as above but to fuse them.(p. 64)"

([**64]** Pratt, 1998, pp. 59-60)

Computer environments may offer enhanced possibilities to build such connection by exploiting the double faced nature of computational objects, which may be both formal an informal:

In particular, microworlds, are based on constructive (as opposed to instructive) representations of pieces of mathematics ([**64]** Pratt, 1998, pp. 60); and, according to Pratt, the building of such representations may sustain the notion of connecting the informal and the formal:

([**64]** Pratt, 1998, pp. 60)

> "The microworld approach would place more emphasis on constructive representations than conventional pedagogies. Indeed the notion of connecting the informal and the formal may lie in the process of building with constructive representations."
>
> ([**64]** Pratt, 1998, pp. 60)

*3.2.5.4.**Dynamic Expression***

The computer environment, differently from paper and pencil environment, is a medium where dynamic representations of the world are possible, together with interactive aspects. As a consequence using the computer is more "engaging to learn then static and abstract formalisms" ([**64]** Pratt, 1998, pp. 60).

Within microworlds a user can interact and control such dynamism, thanks to direct manipulation and programming. The former has the advantage of immediacy, while through the second it is possible to express "fuzzy ideas in a formal, conventional and rigorous languages" and make "ideas become more explicit" ([**64]** Pratt, 1998, pp. 60). Ideas can then be expressed in an observable and rigorous form, which is fundamental in mathematics:

> "It is a fundamental part of mathematics that ideas are expressed in this sort of way and so programming becomes akin to doing mathematics. Teaching the machine becomes a central microworld activity. diSessa has gone as far as proposing that microworlds offer a new form of literacy (diSessa, 1995).
>
> > ...by extending linear language into multiply connected, dynamic, richly textured graphical and interactive forms allowed by computers we may fundamentally extend the material bases for thinking and learning, and with them the whole practice of education.(p. 2)"
>
> ([**64]** Pratt, 1998, pp. 61).

*3.2.5.5.**Purpose and Utility***

According to Pratt, "in conventional approaches to instruction, purpose and utility are often confused" ([**64]** Pratt, 1998, pp. 61). By contrast, in microworld environments, a concept can be used in the form of quasi-concrete object, and as such it has its utility. In his example he talks of the concept of average, and concludes:

> "Because the microworld can provide a quasi-concrete object called average, these utilities may be discriminated without even knowing how to generate average. The advantage may be that such an approach separates purpose and utility, as is usually the case in our everyday learning but in contrast to standard mathematics classroom practice."
>
> ([**64]** Pratt, 1998, pp. 61).

3.2.6. Résumé of significant ideas and problems

In the previous discussion on programming and microworlds, we highlighted that technological systems can be interpreted as potential means for articulation enhancing pupils' expressive power. However, programming and working within a microworld, can be different kinds of activities that are both interesting from an educational point of view. Among their characteristics, and interesting educational aspects, we individuate some which are particularly relevant for our work.

First of all, we highlighted (3.2.3) that the activity of programming computers has the valuable property of offering to the user a new language, which is constrained by the formalism of computers, to talk about mathematics. Such languages, because of their natures, and because of the formal constrains they put, are comparable to formal mathematical language, and as such can be used as an entrance point to the formalism of mathematics. In other words, programming a computer may provide the user with a new language, enhancing his/her expressive power; yet, at the same time, this new language is a mathematical alike one, thus, thoughts that are expressed with it, can be interpreted in mathematical terms, and for this reason offer a possible window to mathematics.

However, in order to produce mathematical discourses, it is not enough to use a mathematical alike language, in fact, also mathematical contents should be involved. This is the case of mathematical microworlds, which embed some mathematical domain, which can be explored by means of the computational objects provided by the environment. Microworlds provide users with tools that correspond to mathematical objects, which can be used on purpose to accomplish tasks within the microworld. A mathematical object is used according to some necessity within the microworld, according to the principle of "purpose and utility" we discussed in section 3.2.5.5. Microworlds provide ready made computational objects, which can be used without knowing how they have been create, or the mathematical principles behind them. Microworlds, according to the principle of "using before knowing" (see 3.2.5.2), give learners the chance to use a mathematical objects (represented by computational objects) before they know them as such. The main idea is that by using computational objects, a pupil may learn about the corresponding mathematical objects.

However, the correspondence between mathematical objects and computational counterparts is a semiotic one, in the sense that computational objects are meant to stay for mathematical objects. As we discussed in section (2.6.), signs have a conventional nature, and the correspondence between their signifying forms and their meanings are built conventionally by communities by means of shared codes of correspondence. Such codes of correspondence, could they be explicit or implicit, have to be shared by a community, if we want the whole community to attribute the same meanings to a given symbol. In other words, the meaning of a sign is not an intrinsic property of its signifying form: a given computational objects, cannot represent any mathematical object, unless this is not established, by convention by a community. As a consequence, if in class practices we want pupils and teachers to attribute the same mathematical meanings to the computational objects of a microworld, then the class must share a code of correspondence defining the semiotic relationship between the microworld and mathematics. To put it in simple words, it may be the case that a teacher interprets a computational object as standing for a mathematical object, while a pupil interprets it merely as a computational object. Furthermore, it may be the case that, even if the pupil interprets the computational object as standing for a mathematical one, the relationship between these objects, as viewed by the pupil, may not coincide to that wished by the teacher.

If the semiotic correspondence between computational objects and mathematical objects is not carefully studied, designed, and implemented in class practices, it may happen that the knowledge learnt by pupils is not consistent with the teacher's intentional mathematical knowledge.

In the case of programming computers, as we discussed, the analogies between computer languages and the mathematical language can be a starting point for building such relationship, at least at a semiotic level. The language of the interaction between the user and the computer shares with the mathematical language at least the property of being a textual language (ex. LOGO microworlds). Such property is lost when we consider visual microworlds based on direct manipulation, where the language of the communication between learner and computer is not even a textual one. In this sense there are less chances that the microworld, without a specific teacher's intervention, evokes to pupils a language based relationship between its objects and mathematical objects. As a consequence, if such a relationship has to be exploited for mathematical educational goals, it has to be the object of a specific teaching/learning process which cannot be demanded only to the microworlds themselves, but need the intervention of the teacher. A possible way to do that, is to exploit the semiotic systems offered by the written and the oral language: such semiotic systems are those used to express mathematical concepts, thus can be used similarly to express concepts related to the considered microworld. In this sense, verbal expression both oral and written, may represent a suitable means to foster the construction of a relationship between the microworld (its objects, its activities, etc.) and mathematics (its concepts, its way of thinking, etc.), and to exploit it for mathematical educational purposes.

3.3. The computer as instrument of semiotic mediation

3.3.1. A channel for communication

According to Mariotti, "mediation" is a quite common term in the literature concerning technology and mathematics education. However, it is often used in an undefined way "just referring to a very vague potentiality of fostering the relation between pupils and mathematical knowledge" (**[51]**, Mariotti, pp 705, 2002).

A possible interpretation of the idea of mediation is that proposed by Noss & Hoyles who take the perspective of communication: the mediation function of the computer is related to the possibility of creating a communication channel between the teacher and the pupil based on a shared language. The language of the interaction, not only opens communication between the user and the machines, but in school practice it may open a channel of communication between the user and the teacher. Consequently, the relationship between the pupils and the teachers can be transformed by the introduction of the computer making communication between them possible . as Noss and Hoyles clearly discussed, such a change of class interaction shapes a shared environment, where communications is possible, and meanings can be expressed, explored and changed:

> "Not the transmission of A's understanding to B, but an arena in which A and B's understandings can be externalised; not a means of displaying A's knowledge for B to see, but a setting in which the emerging knowledge of both can be expressed, changed and explored."
> (**[62]**, Noss & Hoyles, pp. 6, 1996)

3.3.2. Instruments of semiotic mediation

As an historic analysis shows, The development of mathematical knowledge is based on a productive dialectics between theory and practice (**[51]**, Mariotti, pp, 705-706). Key elements of this dialectic relationship are instruments. For instance, in Euclid's Elements a special role has been played by instruments such as the ruler and the compass. On the one hand, they are theoretical products of an effort to rationalise the perception and production of shapes, while on the other hand, they are physical objects, instruments to be used to accomplish tasks, oriented toward the external world. Theories and practices have always nurtured each other, even if sometimes they may have developed independently for a while: their dialectic relationship, was constantly reconstructed with continuous shifts of meaning from one field to the other.

One way to view instruments as nodes of the dialectic between theory and practice, is that of considering them as having a twofold functioning: they can be interpreted on one hand as externally oriented and aimed at accomplishing an action, on the other hand, they can be interpreted as internally oriented, and "aimed at controlling the action" (**[51]**, Mariotti, pp, 2002). Mariotti elaborated such distinction starting from the seminal work of Vygotskij, who introduces the theoretical construct of *semiotic mediation*. In what follows we draw from the elaboration provided by Mariotti.

Vygotskij distinguishes between the function of mediation of *technical tools* and that of *psychological tools* (or *signs* or *tools of semiotic mediation*). They are both part of the cultural heritage of mankind, produced and used by human beings, they evolved in the centuries, maintaining their functions. Although clearly distinguished, 'signs and tools' are assumed by Vygotskij (**[83]**, Vygotskij , pp. 53, 1978) to belong to the same category of mediators.

> "The basic analogy between sign and tools rests on the mediating function that characterises each of them. They may, therefore, from the psychological perspective, be subsumed under the same category. ... of indirect (mediated) activity. "

(ibid. p. 54)

The difference between signs and tools rests on "the different way that they orient the human behaviour". (ibid. p. 54). The function of a technical **tool** is **externally oriented**, it is to serve as the conductor of human activity aimed at mastering nature, whilst a **sign**'s function is **internally oriented,** it is a means of internal activity aimed to master oneself In other words, as it is possible to master the nature, we can also speak of the mastering of oneself; these two kinds of mastering, according to Vygotskij are strictly linked,

> "just as man's alteration of nature alters man's own nature"

(ibid., p. 55).

Externally oriented tools can be converted into internally oriented tools through the *process of internalization*, as described by Vygotskij, thus becoming "psychological tools" and shaping new meanings, "in this sense a tool may function as a semiotic mediator".

Both kinds of tools are an integral part of social activity. Vygotskij himself quotes, besides language, examples of psychological tools from mathematics, as for instance, various systems for counting. What makes a system of counting a tool for semiotic mediation is the fact that it has been produced and employed to evaluate quantity, but at the same time, functions in the solution of problems, so as to organise and control behaviour.

The key idea of this perspective is that of distinguishing between externally and internally oriented tools, and assuming that technical tools, can be converted into psychological ones through the process of internalisation. When this happens, an artefact conceived as a technical tool becomes also a means for mastering oneself. This suggests that instruments can be introduced in teaching practices, in order to exploit the mentioned internalisation process for educational purposes. The question is how this can be done, and what knowledge can be learnt by pupils. When focusing on artefacts, it is important to consider the relationship between how they are conceived and how they are used, and the relationship between their embedded knowledge and their evoked knowledge. On the other hand, Mariotti stresses, for educational purposes, the importance of the relationship between the knowledge evoked by the instrument, when used, and mathematical knowledge:

"an educational perspective requires also to consider the relationship between meanings emerging from the use of the instrument and meanings culturally recognisable as mathematics."

(ibid., pp. 707)

Past and present literature highlights both, the educational potentialities of technological artefacts, but it also shows "the instability of the processes of meanings constructions, related to the use of artefacts" (ibid., pp. 707). According to Mariotti, a Vygotskian perspective can give the possibility to overcome the 'impasse'. When an instrument is introduced into school practice, we can consider two main kind of users, and ways to use them. Suppose a teacher set up activities where the pupils have to use a given instrument to accomplish some tasks; in this case, the teacher herself, is not using the instrument to accomplish the same tasks, but she is somehow using the instrument to teach pupils some intentional knowledge, which is the reason why she introduced the instrument in class practices. Mariotti distinguishes between *instrument* and *instrument of semiotic mediation*:

"[…] as far as the computer is concerned, it intervenes in the activity, but it participates to it in different ways, according to the actors involved in that activity:

- as an **instrument** (the artefact is used according to utilisation schemes); and in that case meanings, if any, may emerge, but the *mathematical* meaning, embedded in it, may remain inaccessible to the user.

- as an **instrument of semiotic mediation**; as the teacher utilises it in order to accomplish communication strategies aimed to develop a specific meaning, related to the mathematics content which constitute the motive of the teaching/learning activity."

(ibid., pp.)

This perspective suggest that the teacher can guide an evolution of the knowledge evoked to pupils by an instrument, but how can such evolution be guided and controlled? A first answer can be found in the following hypothesis stated by the author:

"*Meanings are rooted in the phenomenological experience (actions of the user and feedback of the environment, of which the artefact is a component) but their evolution is achieved by means of social construction in the classroom, under the guidance of the teacher.*"

(ibid., pp.)

The evolution of what pupils learn toward coherency with mathematics, can be achieved through social construction.

To sum up, the key ideas that emerge from the discussion of the approach presented by Mariotti are:

- when a subject participates in an activity involving the use of an instrument, he/she may have access to some knowledge evoked by the artefact, in particular he/she may learn an instrumentally evoked knowledge. Thus a teacher can introduce an instrument in class practices, in order to provoke pupils' learning of a related evoked knowledge;

- this evoked knowledge is not necessarily of the same nature of the teacher's intentional knowledge, but the teacher can guide an ad hoc evolution of the nature of such knowledge, in order to reach consistency with his/her intentional mathematical knowledge;

- a controlled evolution of knowledge can be achieved by means of social construction of a class community knowledge;

- It is the teacher who is in charge of starting and guiding the evolution of such class community knowledge pointing to a coherency with his/her intentional mathematical knowledge.

4. Computers, algebra, and theoretical thinking: refining our research questions

In the previous chapter we gave a general overview of international research concerning technology and mathematics education. In order to situate our research within this panorama, we need to move from the general level to a more specific one, that of computers in algebra education.

4.1. Computers in algebra education, an overview

Research within this field is vast, and it is impossible to account of every study[18], which may be characterised by different: typology of involved technological devices; typology of educational activities; educational aims; pedagogical assumptions; epistemological assumptions.

Nevertheless, by combining such criterions we individuated some main threads that we are going to comment below.

4.1.1. Programming

Computer programming is an activity which has, during some decades, been considered a powerful tool for algebra education, for several reasons, as witnessed by Kieran and Yerushalmy (**[46]** Kieran and Yerushalmy, in press). Within the first studies there was a shared opinion that such activity would aid students in their mathematical problem solving, and it was believed that writing, processing and studying computer algorithms would have promoted the development of mathematical concepts. Though there was no direct link between programming and algebra this link was made within the Logo movement[19] which viewed programming as an algebraic activity because it involved expression of mathematical ideas and processes in general ways and with particular languages and syntaxes. According to Kieran and Yerushalmy, despite its positive results (for instance concerning students understanding of the concepts of algebraic variables and algebraic formalization), the Logo movement "did not go as far as making the connection with conventional algebraic concepts and notation" (**[46]** Kieran and Yerushalmy, pp. 3, in press).

4.1.2. Multiple representations approaches

A widely shared opinion see the effectiveness of computers in the plurality of representation means provided. Mathematical objects can be represented in different ways, included computational representation, and the didactical approaches inspired by this characteristic can be classified as multi representation approaches. Many of the approaches to algebra, within this tread, define themselves as *functional approaches* (**[46]** Kieran and Yerushalmy, in press, pp. 4), and present function oriented approaches to algebra.

For instance, the *VisualMath* curriculum (**[87]**, Yerushalmy & Shtemberg, 2001), is based on a variety of experiences with non symbolic representations of functions, and the dominant conception of letters is of varying quantities. Here we find the idea of studying functions by comparing them, and a particular kind of comparison leads to the study of equations. Within this curriculum there is an attempt to introduce algebraic manipulations by studying graphical and tabular (e.g.. spreadsheet-like) representations of expressions, as well as an attempt to give a sense of the usefulness and purpose of such manipulations (**[46]** Kieran and Yerushalmy, in press, pp. 4).

[18] See for instance appendix (**9.4.5**) of this thesis.

[19] This is another name to indicate the movement originated by Paprrt's ideas on microworlds, and which is based on the use of the Logo language for programming. See chapter **Errore. L'origine riferimento non è stata trovata.**, in particular sections 3.1.2 and 3.2. for more comments on the concept of microworld.

Other, function based, curricula approach algebra through the investigation of modelling problems, like the *Core-Plus Mathematics Project* (CPMP) which uses a variety of linked representations and has as a major theme the study of the connections between function representations (**[40]**, Huntley, Rasmussen, Villarubi, Sangtong, & Fey, 2000). The project is embedded in the Dutch curriculum, and is coherent with the choice of focusing on the interpretation and construction of formulas modelling a phenomenon, describing relationships between quantities (**[46]** Kieran and Yerushalmy, in press, pp. 4-5).

4.1.2.1. ***Using spreadsheets to create meaning for the objects and processes of algebra***

Many representations approaches try to create meaning for the objects and processes of algebra, they seek for tools which enable students to represent experiences as functions. Here we find, on one hand, learning environments such as spreadsheets that allow to manage numerical data and express relationships; on the other hand we find environments based on graphical representations. Both graphs and spreadsheets are interpreted as means for bridging phenomena and algebraic models and methods (ibid., pp. 5-10).

Filloy et al. (**[38]** Filloy, Rojano, and Rubio, 2000) describe their "spreadsheet method" as bridging arithmetic and algebra by involving the construction of solutions to families of problems rather than solutions to single specific problems. They observed students moving from specific examples to the descriptions of general relationships, and assuming the algebraic idea of working with unknowns, as represented by spreadsheet's cells (**[46]** Kieran and Yerushalmy, in press, pp. 4-5).

However, some difficulties, concerning the type of symbolisation underlying spreadsheets are described by Yerushalmy and Chazan: the symbols representing locations in spreadsheet tables are not variables, neither unknowns. Such symbols represent values of the cells, which can change, but as they represent particular locations, they are too particular to be called variables. On the other hand they may be called unknowns because the cells they represent may have no value, but that would sound odd if a value was assigned to a cell (**[86]** Yerushalmy and Chazan, pp. 375, 2002).

Other difficulties are pointed out by Dettori et al. (**[26]** Dettori, Garuti and Lemut, pp. 199, 2000) who point out that the sign of equality, in a spreadsheet, being the assignment of computed values to cells, doesn't allow one to express relationships, making it impossible to handle completely algebraic models. According to Dettori et al. (ibid. 206)., an approach to algebra via functions is limited to the introduction to algebra. In fact if on one hand spreadsheets (which are usually associated to functional approaches) allow the investigation of variation of functions, on the other hand they can cause difficulties concerning algebraic manipulation.

In conclusion, considering both difficulties and potentialities, spreadsheets can be used as a introduction to algebra and its symbols, if imbedded in suitable activity sequences: the following summary (**[46]** Kieran and Yerushalmy, in press, pp. 7) can be presented:

> *Multiple representations*: Spreadsheet environments allow for an integrated use of numerical, graphical, and algebraic representations;
>
> *Requiring predictions*: Predictions in a spreadsheet activity can be made at the initial stage of getting acquainted with the problem situation and at the stage of transition from the numerical table to its graphical representation;
>
> *Generalization by recursion and generalization by position number*: Both methods have their advantages and disadvantages;

Spreadsheet formulas versus algebraic expressions: The difference between the two objects, one belonging to the computer environment and the other to mathematics, is sometimes more than syntactic and can cause some conceptual difficulties;

Lack of transparency: In a spreadsheet table, formulas are “hidden” behind the resulting numbers. As a result, students can encounter both cognitive and technical difficulties in monitoring their work."

*4.1.2.2.**Graphs as bridging language***

Graphs can be used to represent relationships related to either mathematical or non-mathematical situations, thus can be used to bypass algebraic symbols as the sole channel into mathematical representation. They can be used to motivate students to experiment, analyze and reflect on a situation, even when it is too complicated to approach it symbolically. Research on this field has been conducted following Dugdale's mid-1980s initial studies (**[31]**, Dugdale 1993), based on real time generation with probes and Microcomputer-Based Laboratories (**[59]**, Mokros and Tinker, 1987). Working with the tools, offered by such laboratories, implies visual analyses of situations that are different from the ones arising when working with algebraic symbols or numerical tables, here the attention is given to the shapes of the graphs and their qualitative relationships to the situations (**[46]** Kieran and Yerushalmy, in press, pp. 7-8).

Other research threads use the potential of graphs not just to represent relationships between phenomena, but as a source for creating an iconic language to be used to express mathematical ideas. The idea is that of deriving from graphic representation, an hybrid language to be used to bridge the natural language with the mathematical one. Schwartz and Yerushalmy (**[71]**, 1995) propose an intermediate iconic language based on functions and graphs, and related vocabulary, and which includes graphical icons describing how a function and its rate of change may change. A software, *The Function Sketcher*, was created in order to allow mathematical activities to take place in parallel in the iconic and linguistic channels (**[87]**, Yerushalmy & Shtemberg, 1994). As Yerushalmy and Kieran observe, the proposed sign system supported the abstraction of everyday phenomena into a subset of mathematical signs which could be manipulated "with software as semi-concrete objects" (**[46]** Kieran and Yerushalmy, in press, pp. 10).

4.1.3. Dynamic control of algebraic objects

As discussed above particular features of computer environment concern the level of interactivity: Many software environments offer the possibility to directly manipulate their computational objects, by means of peculiar devices such as sliders, slidegraphs, dragging etc. " (**[46]** Kieran and Yerushalmy, in press, pp. 18). In particular, the computational objects, can be representations of mathematical objects, and can be consistently manipulated to grasp (ibid. pp. 18-29):

- *connections between different systems of representations*: for instance by manipulating one kind of representation of the mathematical object and studying the effects of such manipulation on other kind of representations of the same object;

- *properties of the object*: microworld with some dynamic element can be used as an environment where to explore properties of mathematical objects, giving an immediate feedback on changes caused by actions in the microworld. For instance, Zbiek and Heid (**[88]**, 2001) presented an environment, based on a dynamic geometry software, where it was possible to dynamically (by dragging points) change the parameters of an algebraic expression, having an immediate visual feedback; within the system, for instance, special cases of families of functions could be easily highlighted.

- *The relationship between a phenomenon and its mathematical models*: for instance using calculator-based rangers (CBR) to record a phenomenon (e.g. a child walking, or a rolling cylinder) and study the related graphs produced by the calculator (**[1]**, Arzarello and Robutti 2001; **[67]** Radford et Al 2003).

4.1.4. Environments for structured symbolic calculation

Approaches based on multiple-representational environments, interpret symbolic manipulation in terms of relationships between different kinds of representations of a given algebraic expression. Two expressions can be considered equivalent if, for instance, they have the same graphs, or table of values, any transformation of algebraic expression can be studied according to its effects on graphs and tables of values. The technological environments used within such context are characterised by commands directed to the manipulation of different representations of functions (for instance zoom commands), and allow easy switching from one representation to another.

In opposition to multiple representational approaches we find "structured symbolic calculation" approaches, as they are called by Kieran and Yerushalmy (ibid., pp. 30). Structured symbolic calculation is not at odds with the previously described views, but it distinguishes itself for its focus on a particular sign system, the algebraic one. The class of environments used within this approach is characterised by commands that support transformations of algebraic expressions or equations:

- operating on their algebraic structures;
- preserving “equivalencies”.

The support offered can be of different kinds, favouring either simple or complex transformations as in the examples described below.

Thompson and Thompson (**[80]**, 1987) present a study based on the use of EXPRESSIONS, a “special computer program […] that enabled students to manipulate expressions, but which constrained them to acting on expressions only through their structure”. The software represents expressions as trees and the user can operate on expressions by clicking on the action to do (for instance distribute, commute, etc.) and then clicking on the head of the branch of the represented expression that has to be transformed. The proposed activities consisted in transforming a given expression into another given expression. This was done both with numeric expressions and literal expressions. In particular we may observe that the software allowed step by step transformations and included transformation principles based on field properties, such as the properties of neutral elements.

Computer Algebra Systems (CAS), are more complicated environments that allow structured symbolic manipulation, but, as opposed to EXPRESSIONS, they do not always allow step by step transformations. They usually offer commands that transform expressions handling complex transformations condensed in only one step, which become black-box algorithms that, at times, can be "harder to understand then the original problem" (**[46]** Kieran and Yerushalmy, in press, pp. 30). An example of black-box command can be taken from DERIVE, quite a popular CAS: for instance, the command FACTOR returns the factorised form of a given expression, but no hint is given concerning the intermediate steps that leaded to the final expression.

In the described examples it is the software which executes transformations, as commanded by the user, and transformations are in principle algebraically correct. Other environments offer different paradigms of interactions, for instance APPLUSIX (**[13]**, Bouhineau et al, 2002) has a peculiar feature, offering the user to freely transform algebraic expressions, without applying any command. While the user writes the new expression, the software may give immediate, and continuous, feedback on the equivalence relationship between the old and the new expression. This control may be activated or not by the teacher, according to teaching purposes.

4.1.5. Concluding remark

In this section we presented a brief overview on research studies concerning the use of technology in algebra education. We classified the presented approaches mainly in terms of the used technological environments, but also in terms of the different aspects of “algebra” they were focusing on. In order to be able to state our position within this panorama, we may consider a threefold model of algebraic activity (ibid., pp. 34), consisting of "(a) generational activity, (b) transformational activity, and (c) global/meta-level activity". The first involves the creation of meanings for expressions and equations. Transformational activity focuses on the notions of equivalence of expressions and of solution of equations, and involves symbolic calculation. Finally, the global/meta-level activity, concerns all those activities where algebra is used as a tool; in particular it may comprise "problem solving, modelling, noticing structure, justifying, proving and predicting". The examples that we presented are quite well identified by this model: for instance multiple representation technology may concentrate on approaches focusing on generational activity and global/meta-level activity; whilst structured symbolic calculation technology is usually associated with transformational activity. As a consequence, for our research, there is a need to clarify our position, in terms of the algebraic activities we focus on, and in terms of the typology of technology we use.

In the next section we will review some of the main threads of research in algebra education, so that we will be able to state our position concerning the kind of algebraic activity we focus on.

4.2. What algebra?

In the previous section we presented a brief overview of research studies concerning the use of technology in algebra education. We discussed some of the key ideas that can be found in literature, focusing on the underlying on the views of algebra and the typologies of the used technical devices and. It turns out that, coherently with the multifaceted nature of algebra, different researches highlight different aspects of algebra. For instance, we described research studies that concentrate mainly on functional aspects related to algebra, or others that concentrate on symbolic manipulation. In fact, as far as algebra is concerned, there are many aspects which characterise algebra as a subject. Kaput and Blanton argued that "algebraic mature reasoning is a complex composite organized around five interrelated forms, or strands, of reasoning" (**[43]**, Kaput & Blanton, pp. 346-347, 2001):

1. Algebra as Generalizing and Formalizing Patterns & Constrains, in particular, Algebra as Generalized Arithmetic and Quantitative Reasoning;
6. Algebra as Syntactically-Guided Manipulation of Formalism;
7. Algebra as the Study of Structures and Systems Abstracted from Computations and Relations;
8. Algebra as the Study of Functions, Relations, and Joint Variation.
9. Algebra as a Cluster of Modelling and Phenomena-Controlling Languages.

Our study concerns symbolic manipulation, which is related to all the aspects highlighted by Kaput and Blanton. However, being aware of the importance of all the these aspects[20], we focused our research study on the second and the third one, which best fit our view of algebra. In fact, in our perspective, symbolic manipulation is viewed as an activity of transforming expressions by means

[20] In Kaput's and Blanton's words "We need a broader and deeper view of algebra that can provide *school* mathematics and that can support the integration of algebraic reasoning across all grades and all topics" (ibid., pp. 345-346).

of the axioms algebra theory. In the following we are going to present our perspective as opposed to the view of algebra mainly as generalized arithmetic, which is very popular in Italian schools.

4.2.1. Algebra as Generalized Arithmetic

Algebra can be interpreted as generalised arithmetic, in various forms: letter arithmetic; algebra of number patterns generalisation; study of the structure of arithmetic, and the study of letter symbolic expressions with no regard to the meanings of the symbols (**[48]**, Lee, pp. 396, 2001).

The central idea of this approach is that there is a continuity between arithmetic and algebra, which can be exploited for the teaching of algebra once arithmetic is known. However, analysing the connection between Algebra and Arithmetic shows that the relationship between calculation with numbers and calculations with letters is not so direct and transparent. As Lee & Wheeler (**[49]**, Lee & Wheeler, 1989) argued t, in spite of the use of common operation signs, the activities of writing and manipulating expressions in algebra and in arithmetic are quite different. Such difference has been witnessed by several authors who also describe how they can be problematic for pupils' learning of algebra.

According to Bednarz there is evidence of students resistance during the transition to algebra from arithmetic (**[9]**, Bednarz N., pp. 70, 2001). The author suggests that this due to the different nature of algebraic activities and arithmetic activities, leading to problems in algebra learning, when it is introduced simply as a generalised arithmetic. Such problems, stem from the nature of algebra, and not from pupils insufficient mastery of arithmetic. For instance, as described by Henry, obstacles may be individuated as "related to sign systems, means-end habituation, and the related need to gain procedural confidence prior to transitioning to structural understanding" (**[39]**, Henry, pp. 302, 2001).

The different nature can be expressed in terms of a cognitive break, which, according to certain authors consists of the fact that the passage from arithmetic to algebra starts from a modelling process that changes both the nature of "*problem resolution*", and the nature of the "*handled mathematical objects*"; whereas in arithmetic numerical computations are central, the main aim of algebra is to "provide an operative language to represent, analyse and manipulate relations containing both numbers and letters"(**[26]**, Dettori et al., pp. 192, 2000). If the style of arithmetic is essentially procedural, that of algebra is essentially declarative, as it is based on defining and manipulating relations.

According to Thomas, it may be the case that students deal with algebraic expression according to the operations of arithmetic, and this leads to conflicts when trying to give a meaning to "the result" of expressions such as **ab** or $\mathbf{a^2}$ (**[79]**, Thomas and Tall, pp. 583, 2001). Furthermore, such a process oriented approach may lead pupils to interpret expressions, such as **(2x+1)(3x-2)** and $\mathbf{6x^2-x-2}$, as not equivalent because they appear to represent different processes.

From this discussion it emerges that, despite the similarities between algebra and arithmetic, they are different subjects, with different objects and processes, and such differences may cause problems to an approach to algebra as generalised arithmetic.

Such differences can be interpreted in terms of Sfard's *operational-structural* theory (**[73]**, Sfard, 1991). Within such theory, mathematical object is interpreted *operationally* when it is interpreted as a process to be executed, whilst it is interpreted *structurally* when it is viewed as an object. For instance, a numerical expressions, can be interpreted either as a computation procedure to be executed leading to a result, or as an object with its structure and that can be, for instance, compared with other expressions. In the first case we speak of an *operational* interpretation of the given expression, which is viewed as a process, whilst in the latter case we speak of *structural* view (**[74]**, Sfard et al., pp. 193, 1994). Such duality of Algebraic objects (like the Mathematical objects in general) is described by Sfard as being necessary and difficult to be achieved:

> "The formula, with its operational aspect (it contains 'prompts' for actions in form of operators) must be also interpreted as the product of the process it represents."
>
> "[…] our intuition rebels against the operation – structural duality of algebraic symbols, at least initially."
>
> ([**73**], Sfard, pp. 199, 1991)

The operational character of pupils' conceptions related to algebraic formula and expressions tends to persist, thus it is not lost when algebra is introduced, however, it appears to be difficult for pupils to achieve also a structural perspective. In fact, although symbolic manipulations of algebraic expressions are largely present in school practice, the absence of "structural conceptions" appears evident ([**44**], Kieran, pp. 397, 1992).

Two levels of computation can be considered: computation with numbers and computation with letters. there is a "striking difference" between writing and manipulating expressions in algebra or in arithmetic, as Lee and Wheeler clearly point out:

> "In spite of the use of common (operational) signs, what one actually does in the two cases is very different, so different that one cannot be surprised if students do not immediately support the connection. An algebraic expression may perhaps be transformable into equivalent forms, but its value cannot be computed. The same expression with numerical values substituted for the letters is immediately computable and 'collapses', losing all its individual character, into a single numeral".

[…] According to the authors, there is a need for a pedagogy helping students to grasp the connection between algebra and arithmetic, highlighting the "differences between two modes of symbolic behaviour" ([**49**], Lee & Wheeler, pp. 51-52, 1989)

In a previous study ([**20**] and [**56**], Cerulli & Mariotti, 2001 and 2002) we analysed the relationships between these two levels of computation, drawing on the case of a pupil , Francesca, a 15 year old girl from a group of pupils that we interviewed, and that was following an approach to algebra as generalised arithmetic. The study revealed that, contrary to what books and teachers usually state, the transition from computing with numbers and computing with letters is not so smooth: in fact, it may present a cognitive break, as suggested by Francesca: *"Our teacher says that with letters it [computing] is the same as with numbers, but to me it doesn't look the same, it looks quite different [It: non mi sembra la stessa cosa]"*. Francesca, besides her difficulties in manipulating algebraic expressions, somehow individuated a key difference between arithmetic and algebra, as witnessed by her statement: *"[…] Because if I am given 10+3… whilst…[if you give me] a•b+c+d I get stuck… (laughs) I can't work it out."*.

Given a numerical expression, it is always possible to compute a *result*, and that is consistent with the objective of the main activity in Italian school practice, before approaching algebra: given a numerical expression, students are asked to compute its *result*, which is a number. On the other hand, given a literal expression such as that cited by the Francesca, although it is asked to compute, it would be impossible to obtain any *numerical result*. Passing from arithmetic to algebra, the term to compute/calculate (ita.: "calcolo") changes its meaning.

As a matter of fact, grasping the link between computing with numbers and computing with letters, requires a radical change of perspective, at which the properties of the operation itself are the core.

According to Sfard's hypothesis, when computing with algebraic expressions a new operational level must be achieved, but this must be achieved without breaking the link with the previous one. The analysis of Francesca's case shows that not only must the reification of an expression be accomplished (expressions can be acted upon as new objects), and not only must the structural level be consolidated (equivalence between expressions must be stated in terms of their values), but also a relation between the two 'computing procedures' (ita.: "calcoli") must be constructed explicitly.

The key-point is in the change of role that the properties of the operations must achieve: in the case of numerical computation they are basically used to state the equivalence of different computing procedure: two numeric expression are equivalent if after computing give the same numerical result; whilst in the case of literal expression, they have to become *rules of transformation*, i.e. "instruments" to transform one literal expression into another, maintaining their equivalence, this process of transformation can be see as a computation; non result in the numerical sense can be obtained, but a result can be conceived according to the fact that at a certain point no other transformation is possible except invert the relations already used. Consider the example of the distributive property, if we are given the numerical expression **2•(3+4)**, we can transform it computing the sum within brackets, and then we can compute the multiplication, obtaining chain of steps "**2•(3+4)-> 2• 7)->14**"; alternatively we can use the distributive property, and obtain this other chain of steps "**2•(3+4)->2•3+2•4->6+8->14**". The two procedures, of course lead to the same numerical *result*, and for the aim of computing this result, they are equally efficient. Thus, a pupil can chose to use one or the other procedure, and would be able to accomplish the computing task either way, even if he/she knows only one of the two procedures; in other words, for these kind of computations, it is not necessary to know the distributive property. In the case of literal expressions, the direct computation, as clearly expressed by Francesca, cannot be performed, simply because, if we are given **a•(b+c)**, we cannot just compute **b+c**; never the less, something can be done: the expression **a•(b+c)** can be transformed into an equivalent expression using the properties of operations, transformation rules that do not rely directly on numerical computation. In this case the distributive property is necessary if we want to perform a transformation structurally similar to that performed with numbers leading for instance to the chain of steps "**a•(b+c)->a•b+a•c**". In the numerical case, the distributive property it is not a necessary tool for transforming expressions within a computing activity, as it is in the case of literal expressions, and it may be not perceived at all as being an instrument.

In order to be interpreted as instruments for computation, the properties of the operations, must assume a dual meaning (structural and operational): on one hand they state the basic equivalence relations and on the other hand they function as instruments for symbolic manipulation, i.e. instruments by means of which any symbolic transformation is derived (ibid. 231). Within the numerical context, operation properties do not play an operative role; they simply express the equivalence of computing procedures, but they are not necessary, and thus not usually employed for computation. Within the algebraic context, operation properties must assume an operative role and must become the instruments for transforming expressions. Such a change of role is not always made explicit in school practice, and focusing of attention on memorisation of particular shortcut procedures such as algebraic formulas ("prodotti notevoli") may definitely hide crucial point. In conclusion, it seems reasonable to take the hypothesis that this change of role for operations properties should become a first goal in introducing pupils to symbolic manipulation.

4.2.2. Toward a theoretical view of algebra

The above discussion highlighted some educational problems related to interpretation of algebra only as generalized arithmetic, above all, the fact that passing from arithmetic to algebra differences have not to be hidden. Below we are going to suggest a possible approach to algebra which highlights both similarities and differences between arithmetic and algebra. The aim is that of introducing algebra as distinct from arithmetic, although strictly tied to it.

*4.2.2.1.**Moving from arithmetic: algebraic handling of numerical expressions***

The standard arithmetic way to handle numerical expressions is that of computing them, in fact, in arithmetic, expressions are, in general, interpreted computation procedures that are built and executed in order to find a numerical result, corresponding to the solution of the considered problem. As we have seen in section 4.2.1, expressions in algebra (beside the fact that they include literal symbols) are handled in a structural way, as opposed to the operative nature of arithmetical handling of expressions.

A possible way to overcome this cognitive break, is to start by treating algebraically both numbers and operations with them, and thus numerical expressions. If we assume, as a prerequisite to the introduction to algebra, the mastery of arithmetical handling of numbers, operations, and numerical expression. In other words we can think of are suggesting to introducing a new way to handle the same objects, e.g. an algebraic way; but how can this be done?

According to Kaput and Blanton, numbers can be handled algebraically by treating them as placeholders, standing for any number, while operations, can be deliberately left in "'indicated form', unexecuted" (ibid., pp. 347-348). Of course, to handle algebraically a numerical expression is not only a matter of numbers interpreted as placeholders, and operations left unexecuted. By not executing operations we may avoid treating expressions arithmetically, but this is not yet symbolic manipulation: we need to individuate relevant problems and activities, asking algebraic treatments of numerical expressions, problems and activities that can be meaningful either within a literal context or within a numerical context.

*4.2.2.2.**Equivalences***

In a paper concerning the transition from arithmetic to algebra, Kieran describes algebraic *transformational activities*, as being rule-based and including "collecting like terms, factoring, expanding, substituting, solving equations, simplifying expressions, and so on"; At the core of these activities there is the idea of changing the form of an expression, or equation, maintaining equivalencies (Kieran, pp. 123, 2003). In other words, algebraic symbolic manipulation, is based on equivalence relationships, which allow the substitution of a given expression with another one.

In this kind of activity, expressions are to be interpreted according to a structural perspective as opposed to an arithmetic handling of expressions, which happens within a procedural perspective (see also 4.2.1). In such a transition from arithmetic procedural perspective, to the structural algebraic one, a key role is played "by a shift from a procedural view to a relational view of equality" (**[15]**, Carpenter, pp. 156, 2001). In other words, in algebraic symbolic manipulation, the equality sign "=", as to be interpreted as representing an equivalence relationship, while, in arithmetical practice, it is usually interpreted as an input for computing the expression preceding it, such difference is reported to be one of the major blocks when moving from arithmetic to algebra (ibid., pp. 156).

However, the procedural aspects of arithmetic, can be exploited to approach symbolic manipulation within a structural perspective. According to Thomas and Tall, "equivalence is an essential ingredient to understanding the manipulation of symbols", and they propose an approach, called *evaluation algebra*, based on the interpretation of algebraic expressions as processes of evaluation (**[79]**, Thomas and Tall, pp. 592-593). Their idea is that of a computer program which, after substituting to the letters the user's inputs numbers, simply computes the results of given literal expressions,. According to the authors an interpretation of expressions as input-output processes, e.g. according to functional perspective, can favour the building of the idea of equivalence between expressions:

"not only meaning can be given to a range of expressions, but printing the values of expressions such as 2*(A+1) and 2*A+2 will always give the same numerical outputs, allowing the student to sense the equivalencies of these expressions. […] Using the computer in this way may assist students to give meaning to the various ways of writing expressions, including equivalent expressions, which involve different procedures of evaluation yet give the same input-output process."

(ibid., pp. 592-593)

As the authors suggest, this approach has the potential to give meaning to algebraic expressions as process of evaluation, and can lay the foundations to equivalent expressions with different procedures representing the same process.

*4.2.2.3.**Axiomatic algebra***

In summary, a structural perspective is fundamental for symbolic manipulation activities, and it is based on the idea of equivalence, thus also on a relational view of the equals sign. One way to build the idea of equivalence of expressions, as suggested by Thomas and Tall, is that of interpreting them as processes of evaluation, and compare them as input-output processes, thus considering them as equivalent if they give the same results when the same numbers are substituted for letters. However, according to Thomas and Tall, symbolic manipulation can be given flexible meanings only "if the algebraic expressions are seen both as evaluation processes and as manipulable concepts" (ibid., pp. 594). If we pursue these ideas, then, we need ways to manipulate expressions which leave them unchanged in terms of their classes of equivalent evaluation processes. A possible way to approach the algebraic manipulation of expressions, keeping equivalencies, is that of basing the transformations on a set of basic relations of equivalence between calculation processes. In other words, assumed a set of axioms, consisting in the properties of the operations *"symbolic manipulation",* makes sense within a theoretic system and can be interpreted as transforming expressions preserving their equivalence.

Thomas and Tall speak of *Axiomatic algebra*, as a form of algebra having an axiomatic structure based on definitions and deductions (ibid., pp. 594). Such an approach requires a new start with the operations: instead of being interpreted in terms of their arithmetical meanings they must be seen as given concepts the behaviour of which is determined by a list of axiomatic properties. Any property of the operations is interpreted as a "genuine 'law'" acting as a foundation of a theory. Each of such properties is either an explicit axiom has to be deduced from other axioms, "leading to a new deductive form of algebraic structure" (ibid., pp. 595). An approach based on the same idea of interpreting the properties of the operations as axioms of a theory, to be used as means for transforming expressions, was presented in Italy, in the seventies, by Prodi and is described in a textbook for secondary schools (**[65]**, Prodi, 1975).

*4.2.2.4.**Conclusions***

An axiomatic approach, according to the idea of treating numerical expressions algebraically (see 4.2.2.1), can be used also to handle numerical expressions, thus furnishing a bridge between arithmetic and algebra. This bridge functions not only in terms of ways of handling expressions, but also in terms of tasks to be accomplished and problems to be solved. In fact, within an axiomatic theory, the main activity is that of proving theorems, in the case of expression, either numerical or literal, the main activity can be that of proving equivalencies between expressions. Such kinds of tasks and problems are intra-mathematical, and are accomplished and solved by means of symbolic manipulation.

Drawing on the previous discussion, we will define an axiomatic approach to algebra, where axioms are interpreted as means for transforming expressions, thus as means to prove theorems which themselves can become new means for symbolic manipulation activities. In particular, transformations of expressions, by means of axioms and theorems, will be interpreted as an activity of proving statements of equivalence.

4.3. Our approach

4.3.1. Algebra as a theory

Algebra itself is not just an independent branch of mathematics, on the contrary it is often used in order to solve problems originated in other areas. Such activity is basically characterised by the following steps:

10. Translation of the problem into algebraic symbols.
11. Goal oriented manipulation of the obtained expressions.
12. Interpretation of the obtained expressions in terms of the given problem.

The expression "goal oriented manipulation" only makes sense within a problematic context. If that is the case, then the *goal* is to transform expressions into forms that make more evident the solution of the problem, that is a form which gives more information, or a form to which one could apply known theorems or any other piece of knowledge.

For instance, suppose that one obtains from step 1 the expression "x^2**-2x+6-3x**" and suppose that the problem requires one to find the zeros of the expression, then one possible goal of the manipulation could be to factorise the expression in order to be able to apply the rule **"a*b=0 if a=0 or b=0"** to reduce the given problem to that of solving the two equations ("**a=0** and **b=0**"); in this case it could be convenient to transform "x^2**-2x+6-3x** into "**(x-2)(x-3)**", and it is possible to obtain that using twice the distributive law.

Another possible strategy for the solution of the problem could be to apply the well known formula for quadratic equations, in this case one could transform "x^2**-2x+6-3x"** into "x^2**-5x+6**" in order to easily individuate the coefficients needed to use the formula; here the manipulation required concerns the use of the distributive and commutative law.

If the problem just requires us to find out what happens for **"x=0**", then one could just substitute the value in the expression and compute the result, or observe that the only term which wouldn't be zero or multiplied by zero is "**6**".

The example shows how different forms of an expression can lead to different solutions of a problem or can be more suitable for solving certain problems than others. As a consequence it becomes crucial to have tools for transforming expressions, in order to find forms which better help finding the solution of a given problem. The main important tool provided by symbolic manipulation, with this regard, is the notion of *"equivalence relationship between expressions"*, which can be used to transform a given expression into a new one, ensuring the compatibility of the new one with the given problem. In the following we are going to explore such notion.

When we produce an algebraic expression, in order to solve a problem, this expression represents a computational procedure to be used to calculate a result, which is related to the solution of the problem. In such a context, an expression is transformed into a new one representing a new computation procedure, giving the same result starting from the same data. The are allowed transformations should not change the final result. Allowed In other words, from the point of view of the solution of the problem, two expressions can be said to be “equivalent” if, substituting the same data for the letters, the corresponding computation procedures always lead to the same results.

Unfortunately, such a definition of “equivalent expressions” does not provide an effective procedure to establish whether two expressions are equivalent or not. In fact, the substitution process may be non finite. Moreover, such a definition of equivalence can only be used to "verify" that two expressions are equivalent, and it does not help producing a new expression that is equivalent to a given one. Thus a definition of equivalence becomes necessary, that does not necessarily require numerical computations and which can be *used* also *as an instrument to transform expressions*. An equivalence of this kind can be defined taking as axioms the basic properties of numerical operations, and that has been our choice.

In conclusion, we consider *"symbolic manipulation"* as characterised by the activities of goal oriented transformations of expressions using the rules given by the assumed axioms. Thus, symbolic manipulation makes sense within a theoretical system.

Within such a perspective, it is possible to consider the task of comparing expressions, and investigate if they are equivalent or not. The axioms not only tell us when two expressions are equivalent, but, as they can be used to transform expressions, they can become means for proving equivalence relationships: for instance, given two expressions, if one is transformed into the other through a chain of axiom based transformation steps, then this chain can be interpreted as a proof of the equivalence of the two expressions. Furthermore, if an equivalence relationship is proved, it can be assumed to be a theorem, a statement of equivalence that can be used to prove other equivalencies. In other words, from this point of view symbolic manipulation can be interpreted in a theoretical perspective, that is, interpreted in terms of proving statements within a theory, by means of its axioms and by means of previously proved theorems. Of course, the theorems that can be proved, depend either on the admitted types of expressions and on the assumed axioms.

In our research project, we refer to expressions in the sense of polynomials we limit the range of literal expression to the set of polynomials with Integer or Rational coefficients, and the axioms that we assume are the standard axioms of sum and multiplication on such rings. In a dedicated section we will give more details.

Once described/ illustrated/ clarified our perspective on algebra as a theory, we pass to individuate a suitable educational approach compatible with this perspective and based on the use of technology.

4.3.2. A computer based approach to introduce pupils to a theoretical perspective in geometry

Although some authors, as discussed above (see 4.2.2), suggest the possibility of introducing pupils to algebra within an axiomatic perspective, in literature we couldn't find any suitable computer based approaches coherent with this perspective (see 4.1.). Some authors present software based experimentation which focuses on structural symbolic manipulation (see 4.1.4), which in some cases is viewed as based on the properties of the operations as means for transforming expressions, se for instance Thompson and Thompson (**[80]**, Thomposn & Thompson, 1987). Nevertheless, we couldn't find in literature, any approach to symbolic manipulation which takes a theoretical perspective and is at the same time based on some computer environment, or other kinds of technological environments.

However, there are researches that exploit computer environments to introduce pupils to Euclidean geometry from a theoretical perspective. It is the case for instance of the experimentation set up by Mariotti (**[51]**, Mariotti, 2002), where pupils are introduced to the ideas of geometrical constructions and proofs within a theory, and which is based on dynamic geometry software, Cabri Geométre. The approach is based on the theory of semiotic mediation (see 3.3.), and draws on a parallel between the computer environment, (its objects, commands, actions within it) and axiomatic Euclidean geometry. We designed the key idea of our technological approach to axiomatic algebra, drawing from this approach that we are now going to describe.

*4.3.2.1.**Semiotic mediation in the Cabri environment*.**

Mariotti set up and carried out a long term teaching experiment concerning geometrical construction in the "ruler & compass" and the Cabri environments, aiming at introducing pupils to axiomatic Euclidean geometry, and more generally to theoretical thinking.

Given such a didactic problem, the principal motive of classroom activities is centred on the evolution of the idea of proof, and is realised "by means of the evolution of the idea of geometrical construction, within the field of experience of geometrical constructions in the Cabri environment (**[53]**, Mariotti et al, 1997; **[52]**, Mariotti, 2001)". The author introduces the term "field of experience" referring to:

> "the system of three evolutive components (external context; student internal context; teacher internal context), referred to a sector of human culture which the teacher and students can recognise and consider as unitary and homogeneous"
>
> (**[10]**, Boero et al., pp., 1995)

Within the approach presented by Mariotti, there are two main interlaced fields of experience, that of geometrical constructions in the Cabri environment, and that of geometrical constructions in the paper and pencil environment; according to the author, the field of experience can evolve over time thanks to social practices of the classroom. Among such practices, she focuses on the verbal interaction realised by means of "mathematical discussion" as defined by Bartolini Bussi:

> "a polyphony of articulated voices on a mathematical object, that is one of the objects - motives of the teaching - learning activity"
>
> (Bartolini Bussi, 1996).

In the experiment presented by Mariotti, the polyphony of voices concerns a dialectic between the voice of practice, and the voice of theory, that is, a practical conception of graphical construction versus a theoretical conception of geometrical construction (**[51]**, Mariotti, pp. 709, 2002)

In the case of the production of a drawing on a sheet of paper, its validation is demanded by the empirical verification of a practical objective, whilst geometrical constructions have a theoretical meaning overcoming such a practical objective. Geometrical constructions are based on theorems that guarantee theoretical control on the correctness of the procedures followed to realize the constructions themselves. Such theoretical control is not spontaneously achieved, but can be fostered by the activities performed by pupils in the Cabri field of experience:

> “As experimental evidence shows, theoretical control is not spontaneously achieved, but can result from the activities that pupils perform within the chosen field of experience.”
>
> (ibid., pp. 709)

According to Mariotti, the nature of the Cabri environment, may foster a shift from the practical to the theoretical meaning of geometrical constructions, nevertheless the environment itself is not enough, and the intervention of the teacher becomes determinant. However, some elements of the Cabri environment are presented as key elements for the development of a dialectic between the practical and the theoretical level, in fact they can be interpreted as external signs, standing for elements of a geometrical theory:

> “- the primitive commands and macros, realising the geometrical relationship characterising geometrical figures, are the external signs of the basic elements which constitute the theory;

- the dragging function which starts as a perceptual control tool to check the correctness of the construction, then becomes the external sign of the theoretical control."

(ibid., pp. 709)

When geometrical activities are concerned, these elements of the software, are viewed as the external signs on which "the evolution of pupils' internal context is based"; such evolution concerns the development of both, the aimed geometrical theory and at the meaning of theory itself.

In the teaching experiment, presented by Mariotti, the meaning of geometrical construction emerges from activities of construction, within Cabri, and from related mathematical discussions; through the practice of mathematical discussion, the way pupils make sense of the construction activities within Cabri, is elaborated and developed under the guidance of the teacher.

4.3.2.2. ***Construction and use of a theory***

According to Mariotti, two, strictly interwoven, areas of difficulty can be individuated when the idea of proof is concerned. The first is that the idea of validation must be introduced, the second is that the rules for validation must be stated, and their acceptance influence the acceptance of the idea of validation. Within the approach to geometry described by the author, the basic aim was that of introducing pupils to theoretical thinking and in particular to Geometry theory, as a consequence it was decided to build and exploit a dialectic relationship between geometrical theorems and Cabri constructions:

> "Starting in the Cabri environment pupils should have been guided to enter into the geometrical system, the key of access was the link between the logic of Cabri, expressed by its commands, and the Geometry Theory expressed by its axioms and theorems."
>
> (ibid., pp. 715)

However, the software offers many commands, a richness which might foster the ambiguity about intuitive facts and theorems, and constitute an obstacle for pupils to grasp the meaning of proof. As a consequence, in the approach presented by Mariotti, pupils are presented at first with a limited set of commands[21], corresponding to a limited set of Euclidean axioms. Along with the development of the activities, pupils could build their own menus adding commands, corresponding either to axioms of the Euclidean theory, or to new constructions which corresponded to new theorems:

> "Taking advantage of the flexibility of the software environment the microworld was adapted to follow the evolution of the theory: at the beginning, an empty menu was presented and the choice of commands discussed, according to specific statements selected as axioms. Then, in the sequence of the activities, the other elements of the microworld were added, according to new constructions and in parallel with corresponding theorems."
>
> (ibid., pp. 715)

As a consequence, pupils may be guided to slowly build up a geometrical system, in so doing they cope with a complexity they can manage, but at the same time, they participate in the construction of menu and it corresponding axiomatisation. The author reports evidence of the fact that in such kind of experimentation, the construction problem can achieve, for pupils, a theoretical meaning, while the commands of Cabri can be transformed into signs of the theoretical control corresponding to axioms and theorems.

[21] The menus of the software are customisable, thus the teacher can set up the configuration she prefers to persent to pupils.

4.3.3. A computer based approach to introduce pupils to a theoretical perspective in algebra

The project brought forward by Mariotti, which is still in progress, showed how, within the framework of semiotic mediation, a computer environment may offer a support to overcome the well known difficulties related to theoretical perspectives theory (**[53]**, Mariotti et al., 1997; **[55]**, Mariotti & Bartolini, 1998; **[52]**, Mariotti, 2001; **[51]**, Mariotti, 2002).

In the same stream of such project, concerning geometry, we set up a similar experiment concerning introduction to algebra as a theory. We pursued the main ideas of the described approach (see 3.3.2 and 4.3.2), starting from that of using a computer environment as external context for the field of experience related to the activity of proving algebraic equivalencies within a theoretic system. According to our basic hypothesis, symbolic manipulation can be interpreted as constructing a chain of equivalence, corresponding to a deductive chain according to the basic axioms of a theory. Thus we need a microworld which could be a physical counterpart of expressions and axioms and which allowed the user to visualise and make explicit the mathematical entities end relationships which are involved in symbolic manipulation. As for the case of Cabri for geometry theory, we needed a software whose elements could become instruments of semiotic mediation (Vygotskij , 1978), and could be used by the teacher, in the concrete realisation of classroom activity, according to the motive of introducing pupils to symbolic manipulation as a theoretical activity.

4.3.3.1. ***Which technology***

As discussed above, we chose a theoretical perspective where symbolic manipulation is interpreted as an activity of theorem proving within a theory, and where the main involved elements are expressions, axioms (that is the properties of the operations) and the notion of equivalence of expressions. Transformations of expressions are performed by means of axioms, which are, themselves, equivalence relationships. Thus if we want to build a correspondence between the chosen microworld, as well the activities within it, and algebraic expressions as well their symbolic manipulation, we need firstly computational objects which can be interpreted as standing for expressions, axioms, theorems and theories and actions within the microworld, which can be interpreted as standing for transforming expressions, proving theorems, adding theorems to a theory. Such a semiotic correspondence, because of the relative and conventional nature of signs in general (see 2.6.), in order to be established, needs dedicated social classroom practices, and it cannot depend only on the nature of the chosen microworld. However, some microworlds may be better then other, according to the nature of the knowledge they embed, and we individuated some properties that we required from the microworld (**[16]**, Cerulli, pp. 68-70, 1999) which we were going to use, among them we highlight the following:

- *Expressions are computational objects that can be transformed by means of other computational objects to which we will refer as commands*: if symbolic manipulation is interpreted as transformation of expressions by means of equivalence principles, the we want an environment for transforming expressions by means of commands that can be interpreted as equivalence principles.

- *Every transformation of expression, by means of a command, should be invertible, by means of the same command*: this is because when we use an equivalence relationship to transform an expression, the same equivalence relationship can be applied to the new expression to transform it back into the first one. In other words, we assume auto-reversibility to be a key feature of transformation principles based on equivalence relationships, thus also corresponding commands should be auto-reversible.

- *Every transformation command should not execute any implicit transformation if not expressly required by the user*: this is because we want each command to correspond to only one axiom or theorem, and not to a sequence of them.

- *It should be possible to add, and use, new auto-reversible transformation commands, created by the user*: is an auto-reversible transformation command is interpreted as an equivalence relationship, then, when we prove a new equivalence relationship, we prove a theorem, which we want to be able to add to the microworld as a new command.

These three basic requirements oriented our choice toward a quite precise direction, excluding the most popular educational algebra software environments that we cited in this chapter (see 4.1.). For instance, within the stream of multi representations approaches (see 4.1.2), the considered software (for instance spreadsheets) is not conceived for symbolic manipulation, and in general it doesn't even offer commands for transforming expressions and keeping equivalencies. The stream of structured symbolic calculation (see 4.1.4), at first glance, seems to be more promising, because it is mainly based on Computer Algebra Systems (CAS). However most of the used microworlds, even if they present environments for transforming expressions, they do not satisfy the conditions we mentioned above: for instance, the popular software DERIVE has black-box commands, which are not auto reversible, and it doesn't allow users to add new auto-reversible commands[22].

Taking into account the properties of available software, as opposed to our requirements, a new microworld was specially designed, in order to be used to introduce pupils to algebra as a theory, within the framework of semiotic mediation: L'Algebrsita (**[16]**, Cerulli, 1999; **[19]**Cerulli & Mariotti 2000).

In the following we are going to present the educational approach to algebra that we set up, based on L'Algebrista; beside the presentation of the specific teaching experiment, we aim at describing a general theoretical framework concerning the use of microworlds to introduce pupils to theoretical thinking.

[22] For instance, the distributive property is split into two different commands, FACTOR and SOLVE, which are not even exactly one the inverse of the other, and perform many implicit transformations.

5. L'Algebrista: a software to introduce pupils to theoretical thinking and symbolic manipulation

5.1. L'Algebrista: the main ideas underlying it

L'Algebrista is a microworld incorporating the basic theory of algebraic expressions. Activities in the microworld consist of transforming expressions, and a chain of such transformations correspond to the proof of the equivalence of expressions, within a local theory of Algebra expressions.
What follows is a list of the main ideas underling L'Algebrista:

- *A symbolic manipulator which is totally under the user's control.* It is going to be a microworld of algebraic expressions where the user can transform expressions on the basis of the fundamental properties of operations, which stand for the axioms of the local theory.

- *Axioms are represented by the "buttons of the properties of the operations"* which must not have any implicit behaviour; the buttons must realise only transformations which are directly implied by the axiom they represent. Furthermore, a button must not apply recursively an axiom, but only once.

- *Buttons that represent equivalence relationships must be reversible and must include the inverse functionality as well.* This is required to make explicit the meaning of equivalence between expressions, and to associate the correct meaning of equivalence to the "equal" sign ("=").

- *L'Algebrista offers the chance to make explicit the conventionality of Mathematics.* Some buttons will represent or recall conventions of the mathematical community, while other may represent or recall conventions, negotiated within a community of users..

- *The interaction is based on direct manipulation*, using the mouse to select expressions and to click buttons. Thus the user does not have to learn any coding language in order to interact with the system.

- *L'Algebrista is not able to carry out any transformation if it is not guided explicitly by the user using the above mentioned buttons.* In contrast with what happens with other symbolic manipulators, the user has the total control on the transformation activity.

- Once proven, *any new theorem may be represented as a new button and added to the system of axioms and theorems*; thus it can be used to prove other statements and the evolution of the microworld will go on in parallel with the evolution of the theoretical system considered by the user.

5.2. Brief description of the software

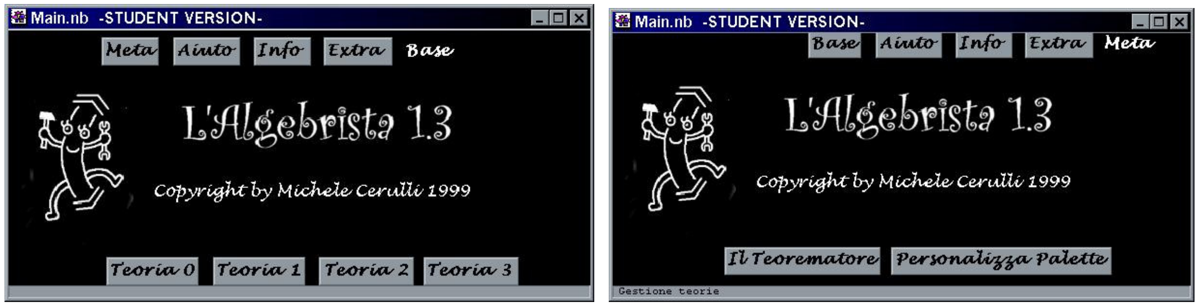

Figure 1 The *Base* menu and the *Meta* menu.

After the start up sequence L'Algebrista offers the user to chose between five different menus: ***Base, Meta, Aiuto*[23]*, Extra*** and ***Info***. As one might imagine, the menus *Info* and *Aiuto* give information concerning L'Algebrista and how to use its commands, facilities and environments. The *Info* menu contains the usual explanations concerning the copyright and licence; the *Aiuto* menu contains information about the main features of the different menus and an "on-line help".

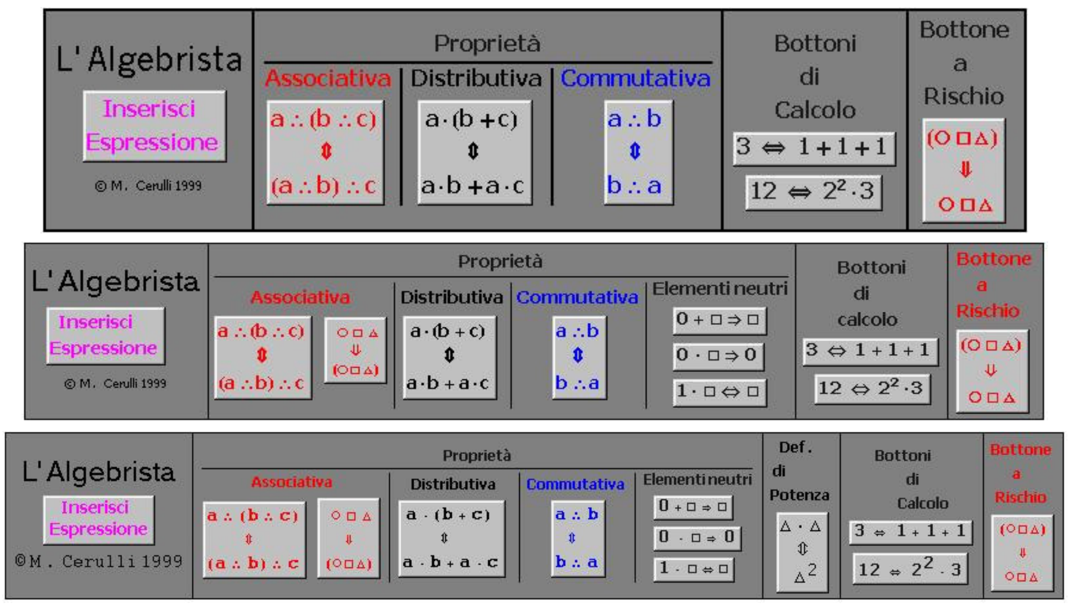

Figure 2 Teoria 0, Teoria 1 and Teoria 2.

5.2.1. The Base menu

This menu introduces the user to the main working environment of L'Algebrista; here the user can chose between several *Teorie* (Theories), i.e. microworlds of algebraic expressions. Each *theory* is made of palettes (windows containing buttons) and notebooks (working environments). In fig. 2 it is possible to see the palettes of the three basic theories we have been using in the classroom experimentation.

To start the activity the user has to write an expression in a notebook and *insert* it into the microworld, using the special button "Inserisci Espressione", then the manipulation of the expression can be carried out by selecting sub-expressions and clicking on the buttons available in

[23] "Aiuto" in Italian means "help". At the moment we only have an Italian version of the software.

the palettes. Each button (except one, as discussed below), always produces expressions which are equivalent (see 4.3.1) to the expressions it is transforming.

5.2.2. The Meta menu

The word *meta* in this case stands for "meta theory", in fact this menu offers two instruments to be used to create new theories[24].

The first instrument is called ***Il Teorematore*** (the theorems maker), it lets the user create new buttons to represent new transformation rules, that can be included in the palettes and used to manipulate expressions in L'Algebrista.

The second instrument is called ***Personalizza Palette*** (Palette personalisation) and is essentially a notebook containing a collection of ready made buttons and instructions concerning the creation of palettes using those buttons and the buttons created with *Il Teorematore*.

With these two instruments teachers and pupils can actually build their own theories, i.e. palettes including buttons referring to selected axioms, definitions and theorems.

5.2.3. The Extra menu

The *Extra menu* is the menu where the palettes produced (see *Meta menu* 5.2.2) by users are located, as distinguished from those furnished by the system. In Figure 1 we present an example of *Extra menu* produced by pupils[25], during one of our class experimentations.

Figure 3 An example of ***Extra menu*** produced by our pupils, it contains palettes for managing powers ("Potenze"), fractions ("Frazioni"), equations ("Equazioni"), theorems in general ("Teoremi").

5.3. Description of the interaction with the basic microworld offered by *L'Algebrista*

Let's now describe the main commands of the theories presented in the *Base* menu analysing some peculiar aspects of the computer-user interaction.

[24] Recall that with *theory* we mean set of axioms, definition and theorems represented by buttons.
[25] Because of technical limitations of the prototype we used, pupils themselves produced each palette, but they were inserted in the extra menu by the author of the software, following the guidelines given by pupils.

Figure 4 shows a palette of *L'Algebrista*; more precisely it represents the first theory we used in our teaching experiments and corresponds to *Teoria 0* in the *Base* menu. This palette is divided into four sectors corresponding to: the button *Inserisci Espressione (Insert expression)*; the buttons of the properties of sum and multiplication; the computation buttons; and the *risky button* ("*Bottone a Rischio"*). This partition is coherent with the distinction between the roles played by each button in the planned classroom activities. In particular, the buttons representing the properties of the operations were separated from the buttons that execute computations, in order to distinguish the activities of transformation based on the axioms, from those based on numerical computations.

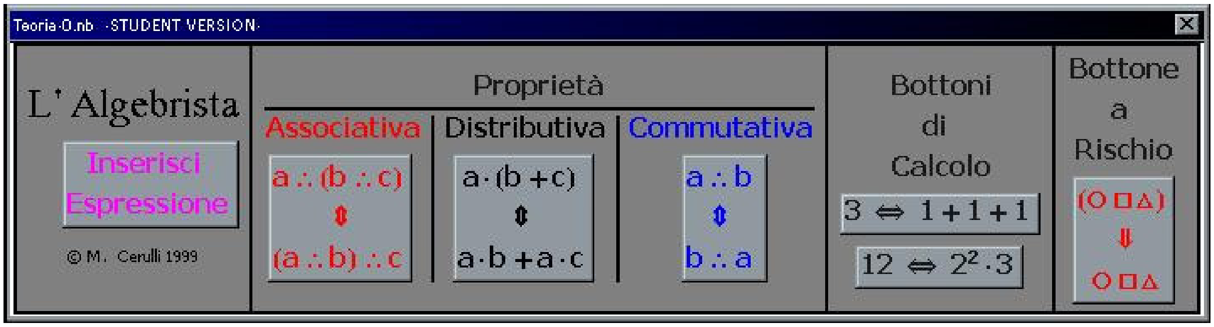

Figure 4 "Teoria 0" (en.: "Theory 0").

An example of interaction is reproduced in Figure 5. The user writes on a blank document the expression he/she wants to work with ("2 * 3 + a 2 - 6" in our example), then he/she selects the expression and clicks *Inserisci Espressione* ("Insert Expression"), thus L'Algebrista creates a new working environment where the original expression is marked on its left with the label *Inizio* ("start").

The operation of *inserting* the expression is fundamental because it proclaims the entrance into the microworld where it is possible to act only using the buttons offered by L'Algebrista.

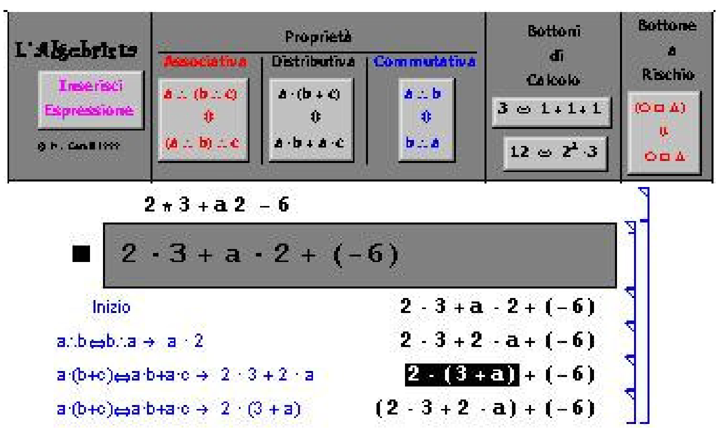

Figure 5 In a notebook the user writes the expression to work with («2 * 3 + a 2 - 6» in the example), then after selecting it the button *Inserisci Espressione* is clicked, thus L'Algebrista creates a new working area where the buttons are active.

We observe that when an expression is *inserted*, its new instance comes out with some changes in its appearence: every multiplication is represented with a dot ("•"), so either stars ("2∗3") or spaces ("a 2") are substituted with a dot ("2•3+a •2"); every subtraction is transformed into sum and every division is transformed into multiplication in the obvious way. L'Algebrista does not know subtraction, and division: this follows from a precise didactical choice because we wanted pupils to work in a “commutative environment”[26].

Interaction always happens by selecting a part of an expression and clicking on a button. The selection tool was designed so that it is not possible to select parts of expressions which are not sub-expressions from an algebraic point of view. For instance, given the expression **a•b+c** it is not possible to select **b+c**, if one tries to select it, the software will automatically extend the selection to **a•b+c**; on the contrary one is allowed to select **a•b** or **c** or **a** etc. This feature corresponds to fact that the expressions of this microworld embed a fundamental algebraic characteristic of mathematical expressions: their tree structure.

Allowed selections	***Not allowed selections***
a•b+c	**a•b+c**
a•b+c	
a•b+c	
a•b+c	
a•b+c	The above selection is automatically extended to the last selection on the left
a•(b+c)	**a•(b+c)**
a•(b+c)	
a•(b+c)	
a•(b+c)	**a•(b+c)**
a•(b+c)	
a•(b+c)	The above selection is automatically extended to the last selection on the left

Table 1 Examples of allowed selections of algebraic sub expressions in L'Algebrista

Going back to the previous example, the expression can be now transformed by selecting the term **a•2** and clicking the ‘commutative property button’; a new expression is produced (written just below, (see Figure 5) , the term **2•a** is substituted by the term **a•2**, while on the left a label indicates the button used and the sub-expression it was applied to. Going forward we transformed one part of the expression using the distributive property, and in the following step, using the same button[27] we inverted the previous transformation. Coherently with our didactical hypothesis, the buttons embed all the functions of the properties of operations without making any particular direction more advantageous. Note that most of the symbolic manipulator use another, different, command in order to invert a specific command.

L'Algebrista's buttons always produce a correct expression that is equivalent to the original expressions to which they had been applied; the only exception is the *Risky Button* which is used to delete parenthesis: for instance it can transform **a+(b+c)** into **a+b+c** but it can also transform **a•(b+c)** into **a•b+c**. This button has been put aside and highlighted, so that the user can distinguish it from the others and use it with particular attention. Its meaning and its use is to be negotiated in the class in order to make clear the conventional use of parenthesis and its relation to algebraic axioms.

[26] A discussion on this choice, and other choices, will be presented in chapter 9. .
[27] The command activated by the button checks the structure of the expression and then decides how to transform it; in case no structure is recognised then the expression is left as it is.

We conclude this section with a couple of observations on the notations used: the commutative and the associative properties have been represented using the symbol "∴", instead of "+" and "•", this is certainly related to a matter of economy, but also it is intended to familiarise students with generalisation of structures.

5.4. “Il Teorematore”: the theorem maker

Il Teorematore is a particular environment which allows the user to create new buttons, fig.4 shows the environment and the available instructions.

The use of *Il Teorematore* is very simple, one just has to write the new transformation rule, to select it, and finally to click on the button *Teorema*. In our opinion it is fundamental that the user does not have to learn any coding language to create new buttons, but he/she just has to use mathematical symbols.

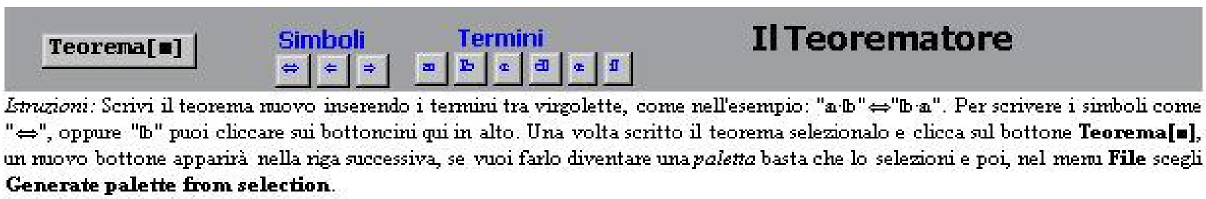

Figure 6 Il Teorematore.

Thanks to *Il Teorematore* the theory embedded in L'Algebrista can grow together with the user's mathematical knowledge. In other words, the user can create as many buttons as he/she wants, and can then use them in his/her future interactions with L'Algebrista.

Coherently with its basic principles, L'Algebrista, thanks to Il Teorematore, lets the user create new buttons corresponding to bi-directional transformations, that is buttons which embed all the functionality of equivalence relationships[28]. This feature strongly differentiates L'Algebrista from most of the popular symbolic manipulators. In particular, DERIVE does not allow the user to create any new command, while other didactical softwares (such as MILO and Theorist) let the user create only mono-directional commands that can be inverted only by using other commands.

Finally we observe that Il Teorematore **does not** check mathematical correctness of a new transformation rule. This is a consequence of a specific didactical choice: we want the pupil to be responsible for the validation of a new theorem or transformation rule. Thus it is the student who will have the control on the set of theorems, i.e. on the theory built up in L'Algebrista. Such a choice makes sense within the peculiar framework of our experimentation, because of the specific didactical contract, but may not be optimal in other cases.

5.5. Technical notes

L'Algebrista, is an application of the more popular software *Mathematica.* In order to run it needs Mathematica 3.0 (or more advanced versions), thus the hardware and software resources needed are the same needed to run *Mathematica*. In particular L’Algebrista is platform independent, within the range of the most popular Operative Systems (e.g. Linux, Macintosh, Windows).

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[28] In particular these buttons behave structurally: if one creates a button corresponding to “$\mathbf{a^2\text{-}b^2=(a+b)(a\text{-}b)}$” then this will apply to “ $\mathbf{a^2\text{-}b^2}$”, and to “$\mathbf{(a+b)\ (a\text{-}b)}$” transforming one into the other, but it will have the same behaviour on more complicated expressions having the same structure; for instance, “$\mathbf{(x+2y)^2 - ((c\text{-}d)^2)^2}$” will be transformed into “$\mathbf{((x+2y) + (c\text{-}d)^2)\ ((x+2y) - (c\text{-}d)^2)}$”.

The code is based on Mathematica language and the application consists of a set of notebooks and palettes.

Each mathematical expression is represented in two ways: “externally”, following the usual mathematical notation, that is infix notation; and “internally”, using prefix notation.

Each button is coded using *Mathematica* graphical features and includes a function; this function translates the selected expression from infix to prefix notation, transforms the expression using a “transformation rule”, and finally translates the new expression into infix notation. Such transformations are all done by the original code produced for the prototype, no algebraic computation is done by *Mathematica* on algebraic expressions, only basic numerical operations (e.g.. sum and multiplication) and numerical factorisation are done by the system.

The function corresponding to a transformation rule does not execute any computation on polynomials, actually the transformation of expressions is based simply on changes of structures. In other words, these functions extract the “leaf terms” from the tree structure of the expression and combine them in a new tree structure.

6. *L'Algebrista* a multifaceted instrument

L'Algebrista is a peculiar software which was conceived as an artefact to be used, by different kinds of subjects, to accomplish several kinds of tasks. Consequently its instrumental embedded knowledge has several different facets, according to the uses *L'Algebrista* was designed for. Here we will focus on the following interpretations of the software as an instrument for given subjects to accomplish given tasks:

- *Operational instrument*: instrument for transforming expressions, proving equivalencies, building and using theories. In this case, subjects using the instrument could be single pupils, groups of pupils, the whole class, or the teacher;

- *Semiotic instrument*: instrument for representing individual and community mathematical knowledge, thus also pupils' mathematical knowledge, either as individuals or as a whole class;

- *Educational instrument*: instrument for teaching-learning symbolic manipulation within a theoretical perspective.

6.1. *L'Algebrista* as a material instrument

In this context, with material instrument we mean an instrument to be used to act upon material objects[29]; *L'Algebrista* is an instrument by which the user operates on mathematical objects (such as algebraic expressions) using mathematical instruments (such as axioms, definitions and theorems expressed as equivalence relationships). However, we may question how a material object, such as *L'Algebrista*, can help a user to act upon mathematical objects, which by their nature are not material[30]. The basic assumption is that there are semiotic relationships associating material objects to mathematical objects: the subject using the instrument interprets the material objects as standing for mathematical objects. In other words, an instrument like *L'Algebrista* operates directly on material objects that the user has to interpret as representations of mathematical objects. The authors of the software designed its objects and commands on purpose, trying to keep a parallel between *L'Algebrista* and the domain of algebra as a theory (see 4.2.2.3). In this section we are going to describe the mathematical knowledge, related to algebra as a theory, that has been embedded in the software.

L'Algebrista presents an environment made of the following kinds of objects (see **Errore. L'origine riferimento non è stata trovata.**):

2. ***Test editor expressions***: expressions written in a text editor environment.
3. ***A button to insert expressions ("Inserisci Espressione" or "Insert Expression") in the manipulation environment***: after the selection of a text-expression it transforms it into a "*L'Algebrista expressions*" located in a manipulation environment.
4. ***Algebrista expressions***: which it is only possible to transform using specific commands of the software;
5. ***Buttons for handling expression***: commands, activated by acting on specific buttons that transform L'Algebrista expressions into other L'Algebrista expressions;
6. ***An environment (Teorematore, or Theorem maker) which provides buttons for that transforming a L'Algebrista expression into a new button for handling expressions***: in other

[29] In the specific case the material objects are elements of the software.
[30] The same observation can be done for any mathematical software.

words this is an enviromnet where it is possible to create new command to be interpreted as new theorems.

It is possible to use buttons (of any kind) on L’Algebrista expressions, without any reference to mathematics, one can transform an L’Algebrista expression into another one using buttons, and can use the Teorematore to transform an expression into a new button. The system itself, to function, does not require any reference to Algebra, and in general to Mathematics, nevertheless its objects embed some mathematical knowledge residing in their properties and functioning. We are now going to analyse in details the mathematical knowledge embedded in the various objects of *L’Algebrista*.

6.1.1. The mathematical knowledge embedded into Algebrista expressions

L’Algebrista expressions were conceived for representing mathematical expressions: they are string of algebraic symbols, the same symbols that are commonly used for algebraic expressions: alphanumerical symbols, brackets, operation symbols, fractions and power symbols.

L'Algebrista embeds mathematical knowledge concerning the algebraic symbols and their use to form algebraic expression.

When the user inserts, or selects, an expression, it has to satisfy two key requirements in order to be accepted by the software for handling it:

- An Algebrista expression has to be made exclusively of the symbols mentioned above, which are the same that are used in algebra, in other words it has to be made of algebraic symbols;
- An Algebrista expression has to be structured compatibly with the rules of algebraic language, for instance the expression "3+ˌ2" would never be accepted even if it is made of acceptable symbols.

In a text editor expression certain symbols are accepted but substituted automatically by the "Insert Expression" button:

- The symbol "^": is commonly used to represent powers in computer environments; expressions like "a^b" are thus automatically transformed into "a^b" according to the standard notation for representing powers. This corresponds to a convention on the mathematical meaning of the symbol "^" in computer environments.

- The symbols "*" and " ": are commonly used in mathematics, and in some computer environments, to represent multiplication; they are transformed automatically into "ˌ", so expressions like "**a*b**" or "**a b**" become "**aˌb**" which in mathematics still represents the multiplication of "a" and "b".

Furthermore, the structure of a Algebrista expression is compatible to that of a corresponding algebraic expression. In other words algebrista expressions, similarly to algebraic expressions, have a tree structure. When the user tries to select a part of an expression, such a structure must be respected, in fact not all the selections are allowed (see 5.3. for details and examples): it is possible to select only parts of the L’Algebrista expression that can be interpreted as sub expressions of the given one from an algebraic point of view.

in summary, L'Algebrista embeds the mathematical knowledge concerning the nature of algebraic expressions: the knowledge concerning the symbols and the sintactical ruels to be used to form algebraic acceptable expressions; the knowledge concerning the structure of an algebraic expression; finally it embeds knowledge concerning conventions on the equivalence of some symbols. Such objects of knowledge are embedded either in the nature of the L’Algebrista expressions, either in the selection system, either in the "Insert Expression" button. The latter embeds some other important objects of knowledge which concerns the relationship between the operators "**+**" and "**-**" and between divisions (and fractions) and powers.

The algebra of L'Algebrista is meant to be a commutative algebra, so that it is possible to use a button corresponding to the commutative property of sum an multiplication, as consequence the microworld was designed for working only with sums and multiplications. However, the user can insert expressions that include subtractions and divisions, but these are automatically transformed into sums and multiplications by the "insert expression" button. In fact, whenever an expression is inserted in the manipulation environment, subtractions are transformed into sums, thus expressions such as "**a-b**" and "**2-3**" become "**a+(-1)ˌb**" and "**2+(-3)**". The "Insert Expression" button thus embeds the knowledge that subtracting an expression from another, is the same as adding to the first expression the opposite of the latter; at the same time it embeds the knowledge that the opposite of an expression is equivalent to the expression multiplied by minus one, "(-1)".

Finally, if an expression in the text editor contains a fraction or a division, then the "Insert Expression" button automatically transforms it in a power by a negative exponent, so "**a/b**" would become "$\mathbf{a}^{\mathbf{-b}}$". Furthermore L'Algebrista includes also a button ("Notation Button") allowing the user to change notation and go back and forth between the two representations of fractions. The two buttons, embed knowledge on the mathematical conventions stating that the two way to represent dividision, as fraction or as negative powers, are equivalent.

6.1.2. The mathematical knowledge embedded into Buttons for handling expressions

The main command buttons (see Figure 2) were conceived as standing for algebraic axioms, theorems and definitions expressed in the form of equivalences between algebraic expressons. Basically they rely on an instrumental conception of the equivalence relationships espressing the basic properties of sum and multiplication, i.e. any equivalence is interpreted as instruments for transforming expressions. In the following section, We are now going to explain in what sense equivalencies can be interpreted as means, i.e.instruments, for transforming one exprtession into another, so that their functioning can be considered instrumentally embedded in the functioning of L'Algebrista buttons.

*6.1.2.1.**The principles of substitution and reversibility***

The basic principle of equivalence relationships in mathematics is that within a class of equivalent mathematical objects, any member of the class can be taken as representative of the others, and such representative can be substituted by an other one at any time. We interpret this possibility of changing representative as a "trasnsformation" of the first object into the second, so that we can speak of an "instrumental function" of equivalencies: working with a mathematical object an equivalence relationship allows one to substitute it with an equivalent one. This is usually done in order to use an objct that better suits our exigencies. Of course, at any time, we can go back to the original representative: a transformation based on an equivalence relationship is always reversible, by means of the same equivalence relationship (see 4.2.2.4 and 4.3.1).

These aspects of an equivalence relationships have been embedded in the basic functioning of L'Algebrista buttons for handling expressions. In fact, each button transforms a given expression into another one, by substituting it, and conversely, the same button can be applied on the second expression in order to transform it back into the first one.

Thus buttons embed the principle of substituting expression with other expressions, and the principle of reversibility of such substituition as in the case of mathematical equivalence relationships. These principles of reversible substitution are embedded in any button for handling expressions, as witnessed and represented also by the formulas written on buttons: each fromula represents two expressions separated by a double left-right arrows (Figure 7). Each of such buttons functions according to an equivalence relationship defining what is the type of transformation it produces, so each of them embeds the relative transformation rules; for instance we have buttons corresponding to associativity, disctributivity and commutativity of operations.

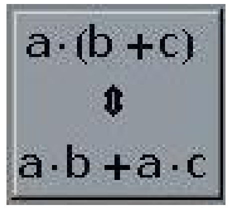

Figure 7 The two expressions represented in the formula written on buttons of equivalence relationships are separated by a double up-down (or left right) arrow.

*6.1.2.2.**Handling the tree structure of expressions***

The equivalence relationships expressed by buttons for handling algebraic expressions are based on the tree structure of expressions: that means that any expression is organized into sub-expressions, consistent with the algebraic sintactical rules. Each sub-expression can be highlited thruough the selection tool, which can be activated by the mouse. This selection tool is controlled so that only “meaningful” expression can be selected. In fact, a characteristic feature is that any button for handling L’Algebrista expressions, in order to function, needs to be applied on a selected expression; the first thing done by the button is to check the tree structure of the selected expression, if this structure matches the conditions embedded in the button, then the button can act and the expression is transformed into another one which matches the conditions for inversion; the button can thus be used to go back and forth from one expression to the other. If a given expression doesn't match the conditions embedded in a given button, then the button simply doesn't apply any transformation, leaving the expression unchanged.

Because it checks the tree structure of the expressions, a button embeds the mathematical principle that in order to apply transformation rules (based on an equivalence relationship) on an object, we first need to verify if the object matches the conditions expressed by the rules, and the principle that the transformation rules do not apply on objects that do not match such conditions.

*6.1.2.3.**Formulas, names and classification***

As already said, the correspondence between algebraic equivalence and L’Algebrista buttons is not limited to the functioning of each button, but it is also extended to the appearance of the button. In fact, the icon which has to be clicked by the mouse has an inscription with the name and the formula representing its associated transformation rule. The formula is always of the kind "**A<=>B**" meaning that whenever the button is applied on an expression with the same structure of A, then the expression is transformed into an equivalent one with the same structure of B; or vice versa an expression with the same structure of B is transformed in an expression of the same structure of A. Of course we could use only the verbal description (the name of the property) or even no inscription at all, and the button would function in the same way.

In the formulas written on the buttons, numerical and alphabetical symbols have different interpretations :. While a numerical symbol represents the number itself, an alphabetical symbol represents a generic term[31], i.e. any well formed algebraic expression. For instance, there is a button with the formula "**A+(-1)•A=0**", such button works on any sum of a term with its opposite, of course it works even on complex terms; thus expressions like "**(2•a+3^b)+(-1)•(2•a+3^b)**", or "**2+(-2)**" would be transformed into "**0**", because their structure is the same as "**A+(-1)•A**". On the other hand the expressions "**a**", or "**3**" would not be transformed at all, because their structure does not match

[31] There are some exceptions as will be shown.

that of "**A+(-1)•A**" whilst, they match the structure of "**0**", but they are not equal to "**0**". In other words, letters in the formula represent variables for expressions, whilst numbers are interpreted as constants.

The formulas written in L'Algebrista are coherent to those commonly used in mathematics, and are accompanied, in *L'Algebrista*, by written text recalling their conventional mathematical names or recalling some of their mathematical characteristicsc. For instance, we can find a button with the formula "**a•(b+c)=a•b+a•c**" associated to the text "**Proprietà distributiva**" (en.: "distributive property").

Buttons are organised into palettes, and distributed in different areas of the palettes according to the mathematical status of the equivalencies they embed, could then be axioms, theorems or definitions of a given theory. For instance, the button of the *distributive property* is situated in a special areas named as "*properties of the operations*", while the button "**a•b^{-1}=a/b**" is situated in the area dedicated to buttons for changing notations. In the same stream, buttons corresponding to theorems are organised in palettes named as "*theroems' palettes*". In other words, L'Algebrista interpreted as a set of buttons, is characterised by a peculiar semiotic system in which names, formulas, colors, and even spatial organisation of the buttons, are coherent to the mathematical contentents they are meant to represent. Different elements of a theory are represented, together with their logical relationships, as we will better describe in the paragraph dedicated to representation of theories.

To sum up, buttons with their formulas and their associated text embed knowledge concerning mathematical notations, and conventions concerning the names, the characterisations and the classifications of the related equivalence relationships.

6.1.2.4."Speaking buttons"

The name *Speaking buttons* (itl.: "Bottoni parlanti") have been coined by our pupils to express the fact that once applied to an expression, in order to transform it, they need some more information. so when one of these buttons is activated, a popup opens and the user has to enter a new expression which will be used for the transformation. The button "**A+(-1)•A=0**" that we mentioned above has such a property, if it is applied to "**0**" a popup opens, and whatever expression is inserted in the popup is used to standing for the "**A**" of the formula; for instance, if "**5**" is inserted, then "**5+(-5)**", or if "**2•a+3**" is inserted then "**2•a+3+(-1)•(2•a+3)**" is obtained. Simple buttons (those that are not *speaking buttons*) transform a given expression into a new one whose terms are built using only the leaves of the tree structure either of the given expression, or of the formula written on the button; in other words new terms are obtained by the old ones simply by recombining their elements together with the elements of the formula of the button. On the other hand, *speaking buttons*, allow the user to insert new elements, new leaves for building the tree structure of the transformed expression.

This kind of buttons embed instrumentally the knowledge concerning the possibility of transforming any expression into an equivalent expression which contains new elements chosen by the user. This, in mathematics, corresponds to commonly used tricks for symbolic manipulation, such as, for instance:

- adding and subtracting a term to a given expression to obtain an equivalent expression
- dividing and multiplying a term by a given expression to obtain an equivalent expression
- adding a term to a given equation to obtain an equivalent equation
- multiplying a term by a given equation to obtain an equivalent equation

6.1.3. The mathematical knowledge embedded into the Teorematore

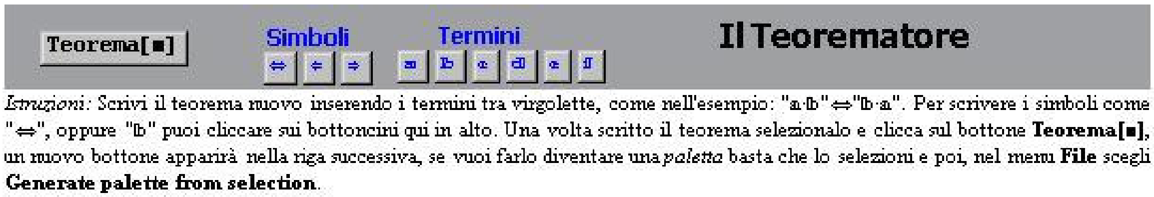

Figure 8 "Il Teorematore".

The "Teorema" (en.: "Theorem") button of the Teorematore (en.: "Theorem Maker") transforms a formula, expressing an equivalence, into a button whose transformation rules are based on the given formula which, itself, is automatically written on the button[32]. Thanks to this transformation of a formula into a button, il Teorematore creates a new instrument that can be used to transform expressions in *L'Algebrista*. Thus the Teorematore embeds the mathematical idea that a statement of equivalence, a formula, can be assumed as instruments and then coherently used as means for transforming expressions.

Looking now at the formulas accepted by the Teorematore, their structure is made of a left term and a right term, separated by a symbol, to be chosen by the "Simboli" available (see Figure 8). The available symbols are a double sided arrow, a left arrow and a right arrow, which implies that it is possible to create bidirectional buttons, but also buttons that are not reversible. This is because the Teorematore can create only simple buttons, whilst it cannot create *speaking buttons*[33]. Thus a speaking button such as "**A+(-1)•A=0**" cannot be created with the Teorematore, but it is possible to create a button of the kind "**A+(-1)•A => 0**" which transforms an expression matching the leftmost term into "**0**", but does not reverse the transformation.

Simple buttons are based on formulas made of alphabetical symbols and numerical symbols, the first are interpreted as variables (thus standing for "any expression"), whilst the latter are interpreted as constants. In the Teorematore there is a clear distinction between *terms* and alphanumerical symbols, the first interpreted as variables standing for "any expression", the latter interpreted as constants. The distinction is stressed by the appearance of the symbols used: they are letteres but written with a special font. The interface presents a list of "Termini" (en.: terms) that can be used to write the formula we wish to state and transform into a button; if we include one such *term* in the formula, the Teorematore interprets it as a term in the algebraic sense of the word, thus in the button produced, any such term stands for a generic expression. On the other hand, if a simple alphanumerical symbol (could it be a letter or a number) is included in the formula, then the button produced will interpret it simply as a constant.

This condition is especially relevant when a produced button checks if a given expression matches the condition for being transformed. Suppose that in the Teorematore we produce a new button using the letter "**a**" and the term "$\mathbb{T}$", for instance we could produce the button "**a•$\mathbb{T}$=$\mathbb{T}$•a**"; if we apply the button on an expression, then it will check its structure, but it will check also if one of the elements of the expression is exactly the letter "**a**", whilst "$\mathbb{T}$" can be any acceptable expression. Thus, if we apply the button to the expression "**x•(2+b)**" nothing will happen, but if we apply the button to the expression "**a•(2+b)**", then it will be transformed into "**(2+b)•a**". In other words the

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[32] However it can be edited by the user.

[33] Actually, this is due to technical limitations of the prototype we used for our experimentation, but in principle this limitation might be ignored.

Teorematore embeds knowledge concerning the fact that letters can be interpreted either as constant or as variables.

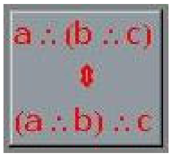
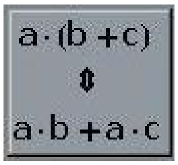
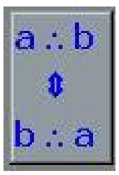
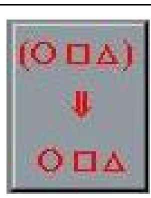
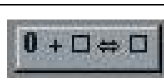
Furthermore, thanks to the use of terms, buttons created by il Teorematore, are generalisations of the formulas used to produce them: in fact, from a given specific formula the Teorematore is able to create a buttons which will work on a whole class of expressions matching the structures of the left and right side of the used formula. In other words, the Teorematore can generalise a given formula into an equivalence relationship based on the structure of the formula, thus incorporating the mathematical idea of generalising algebraic statements on the basis of the structure of formulas.

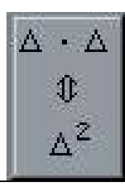
6.1.4. Summary of the buttons of *L'Algebrista* and their embedded knowledge

Besides their general characteristics, and related embedded knowledge, each button for handling expressions embeds its specific mathematical knowledge. Many of the commands now available were created by the authors of the software, while others were created by pupils involved in our experiments, either using the Teorematore or by committing their creation to the authors of the software. in the following table we summarise the set of the main buttoms, we will also indicate who created each button, in order to highlight whose knowledge they embed, be it the knowledge of the class, or the knowledge of mathematicians represented by the author. We will also indicate where each button is situated in *L'Algebrista*, and eventual associated text (see 6.1.2.3).

Button	Associated text	Functionality[34]	Embedded knowledge	Author	Location
	Inserisci Espressione (en.:"Insert Expression")	Insert a text editor expression in the manipulation environment transforming it into an Algebrista expression	Conventional notation and symbology for algebraic expressions with rational coefficients; The subtraction of expressions can be interpreted as the sum of the first and the opposite of the second term of the subtraction; Given an expression **A**, then $\mathbf{-A=(-1)\cdot A}$;	Software creator	Base/teoria0 Base/teoria1 Base/teoria2 Base/teoria3

[34] If not specified, buttons function in bi-directionally, their effect is always reversible by applying the buttons themselves on the obtained expression.

Button	Associated text	Functionality[34]	Embedded knowledge	Author	Location
	"Proprietà associativa" (en.: "Associative property")	Applies associative property of sum or multiplication.	Associative property of sum and multiplication	Software creator	Base/teoria0 Base/teoria1 Base/teoria2 Base/teoria3
	"Proprietà distributiva" (en.: "Distributive property")	Applies distributive property.	Distributive property of multiplication with respect to sum	Software creator	Base/teoria0 Base/teoria1 Base/teoria2 Base/teoria3
	"Proprietà commutativa" (en.: "Commutative property")	Applies commutative property of sum or multiplication.	Commutative property of sum and multiplication	Software creator	Base/teoria0 Base/teoria1 Base/teoria2 Base/teoria3
	"Bottoni di calcolo" (en.: "computation buttons")	Given a sum of numbers it computes the result; given a number it decomposes it into a sum of ones.	Sum of integers Any number can be substituted by its decomposition into a sum of ones, and vice versa.	Software creator	Base/teoria0 Base/teoria1 Base/teoria2 Base/teoria3
	"Bottoni di calcolo" (en.: "computation buttons")	Given a product of numbers it computes the result; given a number it factorises it.	Multiplication of integers; Any number can be substituted buy its factorisation, and vice versa.	Software creator	Base/teoria0 Base/teoria1 Base/teoria2 Base/teoria3
	"Bottone a rischio" (en.: "Risky button")	Removes brackets. It doesn't check the correctness of the operation (see 5.3.)	Associative property of sum and product, which, by convention, allows one to get rid of brackets.	Software creator	Base/teoria0 Base/teoria1 Base/teoria2 Base/teoria3
	"elementi neutro" (en.: "neutral elements")	Adds **0** to the selected expression; or removes a **0** if it	Definition of neutral element of the sum;	Software creator	Base/teoria1 Base/teoria2 Base/teoria3

Button	Associated text	Functionality[34]	Embedded knowledge	Author	Location
		is the left most term of a sum of expressions.	**0** is the neutral element of the sum.		
	"elementi neutro" (en.: "neutral elements")	This is a speaking button. If it is the left most term of a product of expressions is a **0**, then transforms such product into **0**. If only a **0** is selected then it is transformed into a product of **0** and an expression inserted by the user.	Definition of neutral element of the sum; Zero is the neutral element of the sum; The multiplication of any rational number, and any algebraic expression (with rational coefficients) by **0** is equal to **0**	Software creator	Base/teoria1 Base/teoria2 Base/teoria3
	"elementi neutro" (en.: "neutral elements")	Multiplies the selected expression by **1**; or removes a **1** if it is the left most term of a product of expressions.	Definition of neutral element of the multiplication; **1** is the neutral element of the multiplication.	Software creator	Base/teoria1 Base/teoria2 Base/teoria3
	"Def di potenza" (en.: "Definition of power")	Tranforms a power into a product and vice versa.	Definition of power.	Software creator	Base/teoria2 Base/teoria3
	"Il Teorematore" (en.: "The theorem maker")	Transforms two expressions separated by a double arrow into a button to be used to transform expressions.	Instrumental aspects of theorems. A theorem applies to expressions that have a structure compatible to that of the theorem.	Software creator	Il Teorematore

6.2. *L'Algebrista* as a mathematical instrument

As we previously observed, *L'Algebrista* can be interpreted either as an instrument for working with its own material objects, Algebrista expressions and buttons, or as a mathematical instrument for working with mathematical objects such as algebraic expressions, axioms, theorems, definitions and theories. In the latter case the user has to interpret *L'Algebrista* objects as standing for mathematical ones, thus he/she must know a code of correspondences defining how objects, and actions, in *L'Algebrista* can be referred to mathematical objects and actions. A code of this kind was used by the authors of the software who created it according to a web of correspondences which correlate:

- Algebrista expressions **and** algebraic expression
- Algebrista buttons (simple or speaking ones) **and** algebraic equivalence relationships (axioms, theorems and definitions)
- Algebrista palettes of theories (ex.: *Teoria0*, *Teoria1*, etc.) **and** algebraic theories
- A chain of transformations, Transforming an expression A into an expression B **and** a proof that expressions A and B are equivalent.
- Creating new buttons and adding them to a "Teoria" **and** giving to a statement the status of theorem with in a theory.

For a user to interpret *L'Algebrista* as a mathematical instrument, we assume the knowledge of such correspondences to be a prerequisite. In mathematical school practice we aim pupils to interpret L'Algebrista as a mathematical instrument, thus they have to learn a code of that kind. Of course the fact that a software is used during mathematics classes probably suggests to pupils that it is meant to be a mathematical instrument, but this could not be enough if we want them to correctly use it as such and to appropriate of the corresponding mathematical knowledge

In fact inspite of the correspondence between the obejcts and the commands of the microworld domain and the algebraic expressions and their manipulations in the domain of algebra, the two domains are in principle completely independent. The expression "algebraic knowledge inbedded in the microwolrd" remains a metaphor to express the potential link between the two domains, but as all the metaphor needs the use of a consciuos code in order to be effective. Never the less, the fact that the two domains share the system of represetation and the rules of trasformation, leads us to the following didactic assumption

it is possible to exploit such a microwolrd as an instrument of semiotic mediation.

In this sense it is crucial the intervention of the teacher, as the expert who is able to controll the code relating the semiotic system of L'Algebrista and the semiotic system of algebraic expressions. In order to do that, a teacher has to be aware of the possible similarities and differencies between mathematical objects and the objects of the microworld. Thus, in the following we will analyse each of the above mentioned correspondences. We will try to highlight the mathematical knowledge that we think can be evoked by the software itself when used, and the knowledge we believe that needs to be learnt from other sources, thus requiring specific teaching interventions.

6.2.1. L'Algebrista expressions and algebraic expressions

As far as the L'Algebrista expressions are concerned, both their symbols and their structure recall those typical of algebraic expressions. In fact, in the previous sections we talked about some knowledge concerning algebraic expression that is embedded in L'Algebrista expressions. But how can such knowledge be evoked?

First of all, the symbols used to form Algebrista expressions are the same that are commonly used to represent algebraic expressions, thus there is a semiotic code (in the sense described in 2.6.2) of correspondences which associates any expression of the microworld to an algebraic expression; if pupils already know such code, then we can expect them to interpret L'Algebrista expressions as standing for algebraic expressions. If that is the case, we can suppose that what is learnt concerning L'algebrista expressions may result in learning concerning algebraic expressions.

In L'Algebrista, an expression is formed by objects which on the screen are represented as the usual mathematical symbols, but that are really assembled according to a tree structure which is coherent to its structure as algebraic expression. The structure of an expression is not immediately evident to perception, but it can be perceived through the use of the selection tool: when activeted, the selection tool will follow the internal tree structure of the expression. Such tree structure of expressions is functional to the use of the software as instrument for transforming expressions: to transform an expression the first step is always to select part of it, and the only allowed selections are those that correspond to algebraic sub expressions of the given expression (see 5.3.). In other words, the tree structure of algebraic expression, and the conventional precedence rules for computing may be evoked by acting on an expression via the selection tool.

6.2.2. Buttons of *L'Algebrista* and algebraic equivalencies

In school practice we wish pupils to interpret the buttons (simple and speaking ones) of *L'Algebrista* as standing for the algebraic equivalencies they embed. To know and use a semiotic correspondence of this kind is not an automatic consequence of the use of the software, and it may need dedicated teaching interventions. However, the authors of the software deliberaly furnished each button, with a formula and some associated text (see 0 and 6.1.4 for details) that recall the names and formulas commonly associated, in mathematics, to the knowledge that is embedded into a given button (see 6.1.4 for details). These features may evoke the correspondence between a button and its embedded mathematical knowledge.

Let's now suppose that a semiotic relationship has been established, for instance, shared and socialized within the class community, between expressions and buttons in *L'Algebrista*, and algebraic expressions and a set of equivalence relationships. In *L'Algebrista* buttons are *used as instruments to transform* expression, and in case they are interpreted, by the user, as *standing for algebraic equivalencies* we see a potentiality of evoking to the user the idea that equivalence relationships, in algebra, can be *used as instruments for transforming expressions*. Thus *L'Algebrista* embeds, and may evoke, knowledge concerning the instrumental properties function of algebraic equivalencies, which can then be seen not only as a means for stating that two expressions are equivalent, but also as a means for transforming an expression into a new, equivalent, expression.

6.2.3. Working with theories, proving new theorems and using them

The view of algebra that we are considering in this study, considers symbolic manipulation as an activity of proving equivalencies, thus theorems, within given theories (see Figure 2). L'Algebrista was conceived as an instrument to be used for such kind of activities, on the basis of the correspondence between its buttons and the axioms, the definitions, and the possible theorems of given algebraic theories. As we stated, simple and speaking buttons can be interpreted as standing for certain equivalence relationship, which, themselves, can be intepreted as axioms, definitions and theorems of algebraic theories. Given a theory, the first elements to be introduced and used are axioms and definition. they are used as means for proving theorems, according to a set given inference rules. In L'Algebrista the buttons representing axioms and definitions are contained in the palettes called "Teoria0", "Teoria1" etc. (en.:"theory0", "theory1" etc.). For instance Figure 9 shows a palette, called "Teoria2" which includes, among others, buttons corresponding to axioms (the properties of the operations and of neutral elements), and to definitions (definition of power). Such buttons are meant to be used as means for proving equivalences of expressions in L'Algebrista, and are kept as separated from the "bottoni di calcolo" (computation buttons), which are meant to be used to compute numerical calculations. In fact, the software had been designed for distinguishing a practice of computing numerical results of expressions, from a practice of transforming expressions by means of axioms. The first is represented by the computation buttons "bottoni di calcolo", whilst the second one is represented by the other buttons. Such distinction, in our educational approach, is introduced with numerical expressions, and is then overcome passing to literal expressions when the meanings associated to the two practices are merged together in the idea of proving algebraic theorems. In what follows we are going to exemplify in what sense transformations in L'Algebrista can be interpreted as proof of equivalencies of expressions..

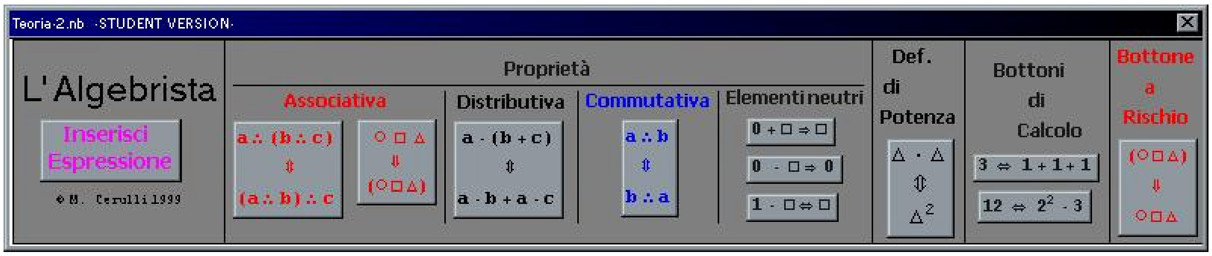


Figure 9 Teoria2, this contains buttons corresponding to axioms such as the basic properties of the operation; it contains a button incorporating the definition of power; it contains buttons for numerical computations.

The buttons represente in Figure 9 can be used to transform one L'Algebrista expression into another one, the sequence of transformation, as it appears on the screen, is shown in the picture below, where the expression $\mathbf{(a+b)^2}$ is transformed into the expression $\mathbf{a^2+2\cdot a\cdot b+b^2}$.

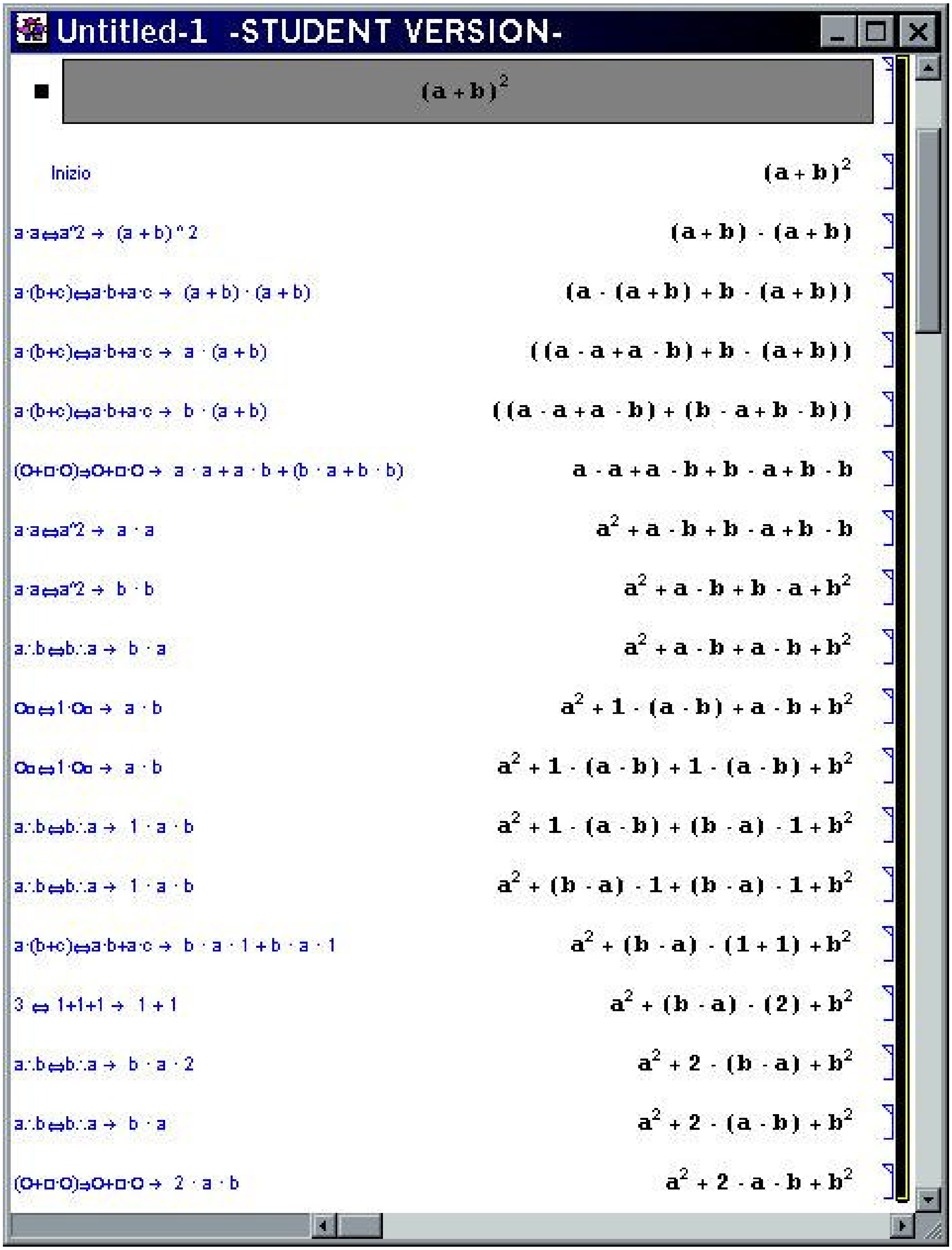


Figure 10 The expression $\mathbf{(a+b)^2}$ is transformed into the expression $\mathbf{a^2+2\cdot a\cdot b+b^2}$ using the buttons of the palette *Teoria2* (see **Figure 9**). This can be interpreted as proof that the two expressions are equivalent according to the axioms of the theory represented by *Teoria2*, which have been used at each step of the transformation process.

Each step of the transformation is obtained by means of a button of *Teoria2*, as indicated by the blue labels on the left (see also description of interactions with the software 5.3.). At this point we can then interpret these buttons as representing the axioms, and definitions[35], of an algebraic theory represented by the palette *Teoria2*, and we can interpret L'Algebrista expressions as algebraic expressions. Thus, the fact that at each step an axiom (stating an equivalence relationship) was used to transform an expression into another one, tell us that we actually built a sequence of expressions which are equivalent in terms of the axioms of the considered theory. As a consequence, ***manipulating*** the algebrista expression $\mathbf{(a+b)^2}$, as shown in Figure 10, can be interpreted as ***proving*** the fact that the algebraic expressions $\mathbf{(a+b)^2}$ and $\mathbf{a^2+2\cdot a\cdot b+b^2}$ are equivalent, within the theory represented by *Teoria2*. The way in which the manipulation is represented, itself, highlights, at each step, the axioms used, coherently with usual representations of mathematical proofs. However, for a pupil to consciously *prove* algebraic equivalencies with *L'Algebrista*, some prerequisites are required, at least he/she has to interpret:

7. *L'Algebrista* objects (expressions, buttons, palettes) as representing algebraic objects (expressions, equivalencies, theories)
8. the manipulation of algebrista expressions (by means of the buttons of a palette) as standing for the manipulation of algebraic expressions (by means of a set of algebraic transformation rules, or equivalencies).
9. the manipulation of algebraic expressions (by means of transformation rules that keep equivalencies) as proving equivalencies of expressions within an algebraic theory (see **4.2.2.3**);

We cannot aspect these intepretations to be evoked by the software itself, rather such interpretaion will constitute the aim of dedicated teaching interventions, in fact the software could be used to transform L'Algebrista expressions without any reference to its possible algebraic counterparts. In order to use L'Algebrista as an instrument for algebraic *proofs*, there is a need to accomplish the interpretations described above, in other words, to activate a semiotic relationship between algebraic proofs and manipulating expressions in L'Algebrista. such semiotic link allows L'Algebrsita to be intepreted as a means to construct/produce mathematical proofs.

Once an equivalence relationship is proved, it can be considered as a new element of the theory that originated it, acquiring the status of theorem which can be used to prove other statements. This in L'Algebrista, is reflected by the fact that, using the *Teorematore* it is possible to create a new button, which can be interpreted as the proved equivalence relationship (see Figure 11), both as a statement and means to be used in a new proving process .

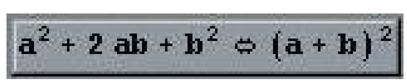


Figure 11 A button created with the *Teorematore* corresponding to the theorem stating that $\mathbf{a^2+2\cdot a\cdot b+b^2=(a+b)^2}$

This new button, is created by the Teorematore and can be placed, by the user, within a new palette of buttons, but not in the "*Teoria#*" palettes, which are meant to represent the initial set of axioms and definition, and cannot be modified; in other words, the particular location of a button in L'Algebrista, can be interpreted in terms of its status within the Theory: being an axiom (or definition) or being a new derived theorem. In picture (Figure 12) we show a palette of theorem buttons, that had been created by our pupils and that include also the button of Figure 11.

[35] See for instance the button corresponding to the definition of power.

Figure 12 A palette of theorem buttons, created by our pupils during an experiment, that includes the theorem button represented in **Figure 11**.

The distinction between the status of axioms, definitions and theorems is embedded in the organization of L'Algebrista in terms of the location of the corresponding buttons: the axiom buttons are placed in the *Teoria#* palettes, within the *Base menu* (see 5.2.1), while theorem buttons are placed in the palettes contained in the *Extra menu* (see 5.2.3).; furthermore, the fact that the *Base menu* offers several *Teoria* that the user has to chose, embed the fact that in mathematics you must always know what is the theory you are working with, and you can chose among several different theories, which have different sets of axioms (which in L'Algebrista's corresponds to palettes having different sets of buttons).

Once a new button is created and situated in a palette, it can be used in future activities to transform expressions and thus to prove new equivalencies. For instance the button that we crated above (see Figure 11) is used in the last step of the proof that $\mathbf{(2+x)^2+10\bullet2+10\bullet x+25}$ is equivalent to $\mathbf{((2+x)+5)^2}$.

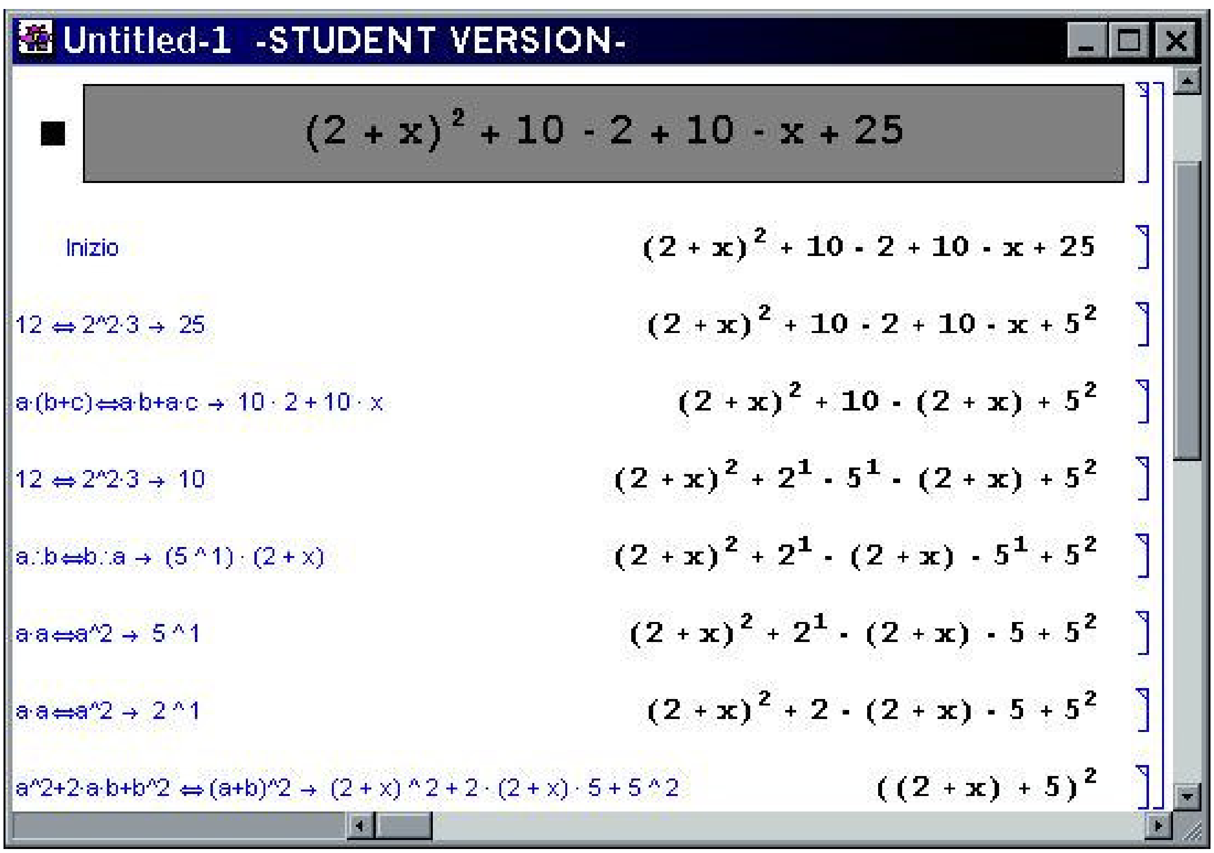


Figure 13 The button that we crated above (see **Figure 11**), incorporating the square of a binomial, is used in the last step of the proof that $\mathbf{(2+x)^2+10\bullet2+10\bullet x+25}$ is equivalent to $\mathbf{((2+x)+5)^2}$.

6.3. *L'Algebrista* as an instrument for teaching

L'Algebrista, was designed, for educational purposes, as a two faceted instrument, on one hand a mathematical instrument (in the sense of being used to accomplish mathematical tasks, as described in 6.2.), on the other hand as an instrument for teaching. But what does an "instrument for teaching" mean? How can it be used? We start from a situation of school practice, where the subjects involved are the pupils and a teacher who is in charge of teaching, thus, when we say "instrument for teaching" we refer to an artefact which is used by the teacher in order to teach his/her intentional knowledge. In other words, when we speak of an instrument for teaching the subject is the teacher, who uses an artefact (In our case *L'Algebrista*) to accomplish his/her teaching task, that is to obtain pupils to learn his/her intentional objects of knowledge. The artificial distinction between teaching and learning is motivated only by the shift on the subject in focus.

Below we are going to describe in what senses the authors of the software conceived it as an instrument for teaching, and in the following chapters we are going to describe how the software is used in our experiment as an instrument for teaching.

6.3.1. An instrument for evoking its embedded knowledge

As discussed in previous chapters, an instrument, when used, can evoke some knowledge to its user; in particular it may evoke its embedded knowledge. A first level for using L'Algebrista as an instrument of teaching, is that of introducing it into school practice, and asking pupils to *use* it, so that its mathematical instrumentally embedded knowledge can be evoked, and learnt by pupils. At this level of analysis, even if L'Algebrista is used only for manipulating its own expression, with no reference to mathematics, we can aspect that some knowledge is evoked and learnt by it users. Such learning outcomes could be related to the teacher's mathematical intentional knowledge, as being instrumentally embedded in the software (see 6.1.), but there is no guarantee that this is the case, (see 6.2.). However, we assume that the user would at least learn how to use the software to accomplish related tasks, such as, for instance, transforming an Algebrista expression into another one. It is possible that the objects of knowledge evoked by this way of using L'Algebrista are compatible with the teacher's intentional knowledge, but we cannot refer to them as mathematical objects of knowledge, unless they are, at least, expressed in a form that would be accepted by the community of mathematicians: in fact, the language of the interaction with L'Algebrista, which consists mainly of selecting objects on the screen, and clicking on buttons, is not the standard mathematical language. At the same time, the practices of manipulating L'Algebrista expressions, are not mathematical practices unless they are not interpreted in terms of corresponding mathematical practices. Here we are stating that because mathematics is the knowledge of a community, the knowledge evoked by the software, in order to be considered as mathematical knowledge, needs to be acceptable as such by mathematicians, either in terms of the performed practices, in terms of contents, and in terms of the form in which they are expressed. The main problem with this usage of the software is that it is not used as an instrument for solving mathematical problems, thus we cannot a priori assume that its evoked knowledge is a mathematical one.

6.3.2. *L'Algebrista* as a mathematical instrument

In a dedicated section (see 6.2.) we explained how *L'Algebrista* can (and was conceived to) be used as a mathematical instrument for accomplishing mathematical tasks, in particular tasks concerning symbolic manipulation and proofs of equivalencies of mathematical expressions. The key idea of this use of the artefact is that the user has to interpret its objects and activities with them, according to a semiotic code of correspondences that associates them to algebraic objects (such as expression, theorems, etc.) and algebraic activities (for instance proving theorems). If this is the case then the mathematical instrumentally embedded knowledge can be evoked. In particular, pupils may learn that in order to prove an equivalence between two expressions, or to manipulate an expression for other purposes, you have to:

- *Choose/know what theory you are working with, and its axioms, definition and theorems*: this may be evoked by the choice of theory palettes and by the palettes themselves, with their contained buttons.

- *Transform expressions according to the axioms, definitions and theorems of the used theory*: this may be evoked by the fact that the only way that you have in which to transform expressions in *L'Algebrista* is to use its buttons which correspond to the element of the theory; in other words, in *L'Algebrista* it is impossible to transform an expression without using an axiom, a definition, or a theorem of a theory.

- *Check, at each step, if the transformation rule that you want to use can be applied on the expression you are working with, which is if the expression's structure is compatible to that expressed by the formula of the used axiom, theorem or definition*: this may be evoked either by the selection procedure of *L'Algebrista*, either by the fact that if the transformation rule embedded in the button does not apply on the expression, the button leaves the expression unchanged.

Furthermore, we have been talking of axioms, definitions, theorems, theories and proofs, which are characteristics of all the branches of mathematics, and *L'Algebrista* may evoke some of their characterisations which are valid in mathematics in general. For instance, the above list can be generalised to:

- *Choose/know what theory you are working with, and its axioms, definition and theorems.*

- *Prove theorems according to the axioms, definitions and theorems of the used theory.*

- *Check, at each step of a deduction, if the principle that you want to use can be applied, that is, check if the hypothesis of the used axiom or theorem are verified.*

6.3.3. Building and Exploiting the semiotic correspondences

The authors of the software hypothesised that mathematical knowledge can be evoked when *L'Algebrista* is used as a mathematical instrument within mathematical activities, as described in previous sections. However, as we already explained, a requirement for this to happen is that the user interprets *L'Algebrista* in mathematical terms, thus referring to its objects as if to mathematical objects and to activities with it, as if to mathematical activities. In order for this to happen, a semiotic code of correspondances between *L'Algebrista* and mathematics is needed. Semiotic codes, by their nature, are conventionally built and shared by the communities of the persons using them (see 2.6.2). Of course we could consider the extreme case of a community of a single person, but in our case, we are interested in communities of at least two persons, a pupil and a teacher. If the teacher has to use a software (*L'Algebrista* in our case) as a teaching instrument, by introducing it in class practices as a mathematical instrument, then, the pupils as well, have to interpret the considered artefact as a *mathematical* instruments, in a way which is compatible to the teacher's way. For our work, we assume that the semiotic code of correspondences has to be shared by the whole community of the class, included the teacher, otherwise, there is no guarantee that pupils interpret *L'Algebrista* as a mathematical instrument in the ways wished by the teacher.

To build a semiotic correspondence between two objects, means either to use one as a sign for the other, either using a third signifying form as a sign for both of them. In the following, we will illustrate how written symbols, in paper and pencil environments, can be used at the same time as signs for both mathematical objects and *L'Algebrista* objects, thus fostering a semiotic link between the two kinds of objects. Such an approach was embedded in *L'Algebrista* as a teaching instrument in the sense that the software was conceived so that any of its elements are represented with symbols compatible to standard written mathematical symbols (see 6.1.2.3).

In the educational approach for which L'Algebrista was designed, mathematical meanings are considered as rooted in the practice with the microworld. Their evolution is guided by the teacher by means of communication strategies which are based on the idea of linking semiotically mathematical meanings to the practice of L'Algebrista. Such links are obtained by means of the introduction of mathematical words as referred to L'Algebrista's objects and practices. Thus the teacher introduces words such as *proof, axiom, theorem, definition, expression, theory*. A key hypothesis which was behind the creation of *L'Algebrista* is that even if pupils may not know the mathematical meanings of the words *proofs, axioms, theorems, definitions, expressions, theories*, it is possible to use such words from the beginning, to refer either to *L'Algebrista*, or to Mathematics, in this way the sphere of practice furnished by *L'Algebrista* may contribute to the building of the mathematical meanings of such words. At the same time, we hypothesised that, because *L'Algebrista is not* Mathematics, there is a need to distinguish clearly between the two domains, and mathematical learning outcomes may be derived not only from analogies, but also from differences. Further on we will show how we tried to build and exploit these webs of relationships between Mathematics and *L'Algebrista* in our classroom experiment.

7. The experiments: definition of teaching/learning paradigm

The research project we are going to report on, started some years ago within the framework of a long term teaching experiment, which is to be considered to be a "research for innovation": action in the classroom is both a means and a result of the evolution of research analysis (**[7]**, Bartolini Bussi, pp. 1, 1998). One of the main objective was to investigate the feasibility of a teaching approach centred on the use of microworlds (Cabri-Géomètre and *L'Algebrista*), and aimed at developing theoretical thinking in both geometry and algebra (for the case of geometry see **[52]** and [51], Mariotti, 2001 and 2002).

Our main concern here is to present a paradigm for putting into practice our educational approach, however, this paradigm is to be considered a result of our research analysis and its evolution interlaced with classroom practice. Thus, we will begin by presenting the main motives of the experiment, and end up by presenting the paradigm itself. In the following chapters we will exemplify and discuss the paradigm, on the basis of collected data.

7.1. A pilot study

A pilot study was conducted, revealing that an approach in the framework of semiotic mediation, based on *L'Algebrista*, could give positive results in terms of introduction of pupils to a theoretical perspective and to symbolic manipulation ([16], Cerulli, 1999; [19]Cerulli & Mariotti 2000). A detailed description of this pilot study is beyond the scope of this thesis, and can be found in ([16], Cerulli, 1999), here we will briefly present it and some results that we find relevant for the discussion of this thesis.

7.1.1. The experiments

The first experiment was a medium term one, conducted in the school year 1998/1999 from October 1998, to February 1999, with pupils of a "classe I of a Liceo Scientifico", that is ninth grade pupils within a school oriented toward scientific studies. The scholastic background of the pupils included computations with numerical expressions, but not management of literal expressions; our intervention concerning the introduction of pupils to symbolic manipulation, fitted the institutional curricula, both in terms of contents and in terms of timing.

The experiment aimed at verifying the effective usability of the software in classroom practice, its prototype nature couldn't guarantee perfect functioning, and we supposed pupils could give use indications for improving the microworld. As a consequence the author personally assisted to all the lessons, acting mainly as "*the developer of the software*", but often participating actively in the teaching/learning process. For these reasons, the experimental environment cannot be considered as that of the normal interaction environment between teacher and pupils.

The sequence of educational activities was mainly organized in three phases: a preliminary phase aiming at highlighting pupils personal views the symbol "=" and of the activity of calculating with numerical expressions; a phase of introduction of manipulation of numerical expressions by means of the properties of operations; a final phase of introduction of literal expression, and their manipulation, still in terms of the properties of the operations. The activities proposed were centred on the task of comparing expressions and the educational motive was that of developing the meanigs of equivalence of expressions and proof of such equivalence within an algebraic theory; proofs were done either within L'Algebrista, by means of buttons corresponding to the properties of the operations, either within paper and pencil environment, with the requirement of always stating the used property of the operations. Some of proved equivalences, according to the decision of the class, were given the status of new theorems, and new corresponding buttons of L'Algebrista were created using the Teorematore (en.: "theorem maker", see 5.4.). More details on the rationale of educational sequence will be given in a section dedicated to the latest experimentations, where we will describe the milestones of the sequences of proposed activities, reporting how such sequence evolved from the first experimentation.

7.1.2. Indications from the pilot study

The first main indication that we received from our pilot study, was a positive feedback concerning the generic usability, and effectiveness of the software in school practice. Moreover beside such encouraging indications, we got positive feedback concerning our entire educational approach to symbolic manipulation. Data showed evidence that pupils reacted positively when faced to new situations, using appropriately the algebraic tools they had previously experienced (ibid., pp. 57-60); moreover, they reacted positively to activities of proving and using new theorems (ibid., pp. 60-61).

One result in particular was that pupils moved from a procedural perspective to a structural perspective. This evolution concerned mainly the interpretation of the structures of the manipulated expressions, and the interpretation of the properties of the operations (ibid., pp. 50-56). Pupils began to interpret the properties of the operations as bidirectional tools to be used to manipulate expressions, and not just as rules for computing the results of the expressions, as the preliminary test showed their starting position to be. Such an evolution seemed to be related to two main reasons: the features of the microworld (mainly invertibility of buttons, and the selection tool, see 5.3. , 0, 6.1.2.2), the particular appraoch to symbolic manipulation based on the notion of equivalence of expressions and consequently, based on the task of comparing expressions (see 4.3.1 and 4.3.3).

However, beside the positive and encouraging indications, the experiment gave us important feedback in terms of suggestions for the following experiments. We obtained mainly three kinds of indications:

- Indications on how to improve the software, both in terms of interface and in terms of features to be added.
- Indications on the types and timings of the activities.
- Indications on the general structure of the whole teaching/learning process

Details of the first two kinds of feedback will be given later; where the last is concerned we would like to observe that although the actual sequence of activities basically reflected our plans, we could observe some crucial variations. In particular, class discussions were not developed as much as we wished, failing to develop the idea of "mathematical discussion" (see 4.3.2.1). Furthermore in parallel to computer laboratory activities, we had planned complementary class activities (with no computers); unfortunately such activities were not developed, thus the experiment was brought forward only in the context of the computer laboratory.

In fact the phase of setting up of the experiment, and putting theory into practice, was found to be partially faulty, and highlighted a need to better define our educational approach, in order to better share it with the teachers involved in the experiment. The difficulties with the teacher in charge of bringing forward the experimentation were, at least partially, solved from the second year of experimentation, which was conducted by an experienced teacher, belonging to a research group and already involved in a similar project conducted by Mariotti (see 4.3.2).

7.2. A long term teaching experiment

On the basis of the results and indications drawn from the study project, a new experiment was set up, involving two classes one for the school years 1999/2000 and 2000/2001 and the other for the school years 2000/2001 and 2001/2002. Both groups of pupils started the experiment at the beginning of grade 9, and ended it during grade 10.

As previously discussed, our experiment was conceived as a counterpart, for algebra teaching, of an experiment set up in previous years by Mariotti, concerning the introduction of pupils to geometrical constructions and theoretical perspective, using the microworld Cabri (see 4.3.2). Our pilot study showed the feasibility of the proposed approach, and its positive results, also in terms of the introduction of pupils to theoretical thinking ([16], Cerulli, 1999; [19], Cerulli & Mariotti, 2000). However it revealed a generic difficulty in sharing with the teacher involved the basic pedagogical and psychological assumtions / theoretical framework inspiring our approach, in particular some key ideas and principles were not actually put in practice (see 7.1.2), and we considered our approach as being only partially tested; as a consequence for the second experiment we chose to work with a teacher who had previously been involved for some years in the Cabri project, and was familiar with the approach and the put in practice of its the key principles. Moreover, this choice allowed us to experiment the approach simultaneously in the cases of algebra and geometry, with the same pupils and the same teacher; in fact the Cabri project and the *L'Algebrista* project, were conducted in parallel, with the joint educational goal of introducing pupils to theoretical thinking, both within the context of geometry and the context of algebra.

In such a way our approach could be experimented entirely, allowing us both to test and improve it, pointing toward a clearer definitions of its theoretical and practical principles in order to make it more shareable within the community of educators (either researchers or teachers).

The experiment was conducted, in class, by the teacher, and we personally assisted in activities that we considered to be central, taking notes and audio recording class discussions, and collecting pupils written protocols; however, thanks to the teacher we also collected protocols of activities that we did not assist in. The conduction of the experiment was planned and reviewed through a weekly meeting with the teacher who reported on the activities to us and to other teachers involved in the Cabri project, participating in the meeting. Thanks to this cycle of meetings, and to our presence in class, we were able to adjust and define the approach *in itinere*, either in terms of educational strategies, or in terms of the activities proposed to pupils. This work had been particularly intense in the first two years of this experiment, concerning mainly the first of the two participating groups of pupils; consequently, the teacher was confident enough to conduct the experiment with the second group of pupils in complete autonomy, but she kept reporting to us on its development, and we kept assisting to some key activities and collecting data. This gave us positive feedback on the practicability of our approach in a context independent from the presence of researchers, thus we undertook a revision of the whole experiment, and taking into account the collected data, and the evolution *in itinere* of the experiment itself, we defined a paradigm for putting into practice an educational approach to theoretical thinking using a microworld.

7.3. Building theories: working with microworlds and writing the mathematical notebook

Our educational approach, being it developed on long term experiments, can be described both at the macro level of the general organization of the sequence of activities, and at the micro level of the teacher's management of the single types of activities.

In this section we are going to describe the general scheme of the cyclic structure of the set of activities that we propose. Such a structure is described in terms of the theoretical framework of semiotic mediation, and the main types of proposed activities will be defined. In the following chapters we will exemplify this structure showing examples of the types of activities that we are defining below. On the basis of the presented exemplifications, we will then describe some key ideas of the management, by the teacher, of the main types of activities, according to the framework of semiotic mediation.

7.3.1. Microworlds and semiotic mediation

As previously discussed (see 3.3.), the *notion of semiotic mediation* is central within a vygotskian theoretical framework. Given an artefact, it can be used by the teacher to exploit communication strategies aimed at guiding the evolution of meanings within the class community; this can also be the case for the computer which can be used by the teacher in order to direct the learner in the construction of meanings that are mathematically consistent ([51], Mariotti 2002).

Our approach (see 4.3.2 and 4.3.3) is based on the general hypothesis: "Meanings are rooted in the phenomenological experience (actions of the user and feedback of the environment, of which the artefact is a component), but their evolution is achieved by means of social construction in the classroom, under the guidance of the teacher" (ibid.). Thus an artefact can be a source for the construction of meanings by its users, but consistency with Mathematics is not a priori guaranteed and needs to be built under the guidance of the teacher. As a consequence, activities within a microworld need to be interlaced with other social activities guided by the teacher in order to reach the construction of the mathematical meanings she is aiming to.

Based on these assumptions our approach is organised in the following cycle of activities:

10. Problem solving activities within the microworlds: this is the field of phenomenological experience where we assumed meanings to be rooted.
11. Problem solving activities in the paper and pencil environment: this is the standard environment for developing mathematical activities, thus we exploit it as a junction point between mathematical activities and activities in the microworld.
12. Production of reports (written or oral) concerning problem solving activities: students' experience is fixed into signs on which collective discussion will be based.
13. Collective discussions, i.e. Mathematical Discussions (see 0): starting from the reports produced by the pupils, the teacher tries to guide the class in the construction of socially shared meanings, consistent with didactical aims.
14. Production of reports concerning collective discussions: the results achieved in collective discussions become part of the class culture, and as such are expressed and fixed into written text, that may serve as a basis for future activities.

This cycle describes the general structure of the teaching sequence and focuses on the main aspects we want to discuss in this thesis. In particular the articulation between experiences centred on activities within the microworld and semiotic activities, based on collective discussion and production of texts.

7.3.2. Working in a microworld

The two microworlds (Cabri and L'Algebrista) share interesting features which, according to the shared Vygotskian framework, are similarly exploited in both the Geometry and the Algebra teaching experiments.

15. Objects and commands can be thought of as external signs of the fundamental elements of a corresponding mathematical theory (Geometry or Algebra).

For instance, basic *tools* are signs of *axioms* and *definitions of a Theory*; *new tools* may be introduced using a specific command (Macro construction in Cabri, Il Teorematore – i.e. Theorem Maker in *L'Algebrista*); these new commands become signs of *theorems*;

16. actions within the microworld correspond to fundamental metatheoretical actions, concerning the construction of a theory.

For instance, *adding new buttons* to those already available corresponds to the meta-theoretical operation of *adding new theorems* to a theory. In the case of Cabri it is possible to create macros (and add the corresponding commands to the menu) that synthesise geometrical constructions and that can be used at any moment. In the case of *L'Algebrista* it is possible to create new buttons representing equivalence relationships between algebraic expressions and that can be used at any moment by the user in order to transform an expression into another one.

Due to the described feature (for more details in the case of L'Algebrsita see chapter 5. , while for Cabri see [52], Mariotti, 2001) L'Algebrista and Cabri result to be good potential environments for phenomenological experiences concerning the production and the use of theorems. Furthermore, they offer the possibility to experience the act of adding commands to the software. In other terms, once a semiotic link with mathematics is built (for a preliminary discussion on this point see [18], Cerulli, in press), the two microworlds make it possible to directly experience the development of mathematical theories by proving and adding theorems, through the effective operations of creating and adding new commands (see 4.3.3.1, 5.4. , and 6.2.3).

7.3.3. Working in the paper and pencil environment

The paper and pencil environment is both the environment where pupils developed most of previous experiences with numerical and litteral expressions , and in any case, paper and pencil is the standard environment for developing mathematical experiences. In this sense, it can be used as a junction point between pupils' past experience of mathematical practices, and new practices in the microworld. By setting up similar activities with common goals in the two environments it is possible to introduce pupils' to two different kinds of practices aiming at accomplishing the same mathematical taks: the practice of the microworld, and the practice of the paper and pencil environment. These practices are distinct because they take place in different environments, but they share a common mathematical motive. The practice within the microworld is completely new for pupils, whilst they are familiar with some mathematical practices in paper and pencil. The meanings originated within the microworld are imported in the paper and pencil environment, and evolve, under the guidance of the teacher, toward new mathematical meanings expressed in the paper and pencil environment, the standard environment for expressing mathematical meanings. Such evolution is guided, by the teacher, exploiting the complex relationship between the two environments and related practices, in the following chapters we are going to discuss further on on this point by means of examples taken from our experimentation.

7.3.4. A notebook as a representation of the culture and the history of the class

Together with the microworld, L'algebrista[36], another specific tool characterises our experimentation: the *notebook* (ital. "quaderno di classe"). Each pupil is asked to edit a personal *notebook* where any result, discussed and socially accepted in the class, will be reported and officially recognized as a piece of mathematics. In particular, each notebook contains the updated list of the axioms and theorems (either in algebra or geometry) of the theory the class is working with, and when a new theorem is produced it is added to the list. Thus the notebook is a personal enterprise, but may be considered a representative of the culture and the history of the class, where the elements of the theory are fixed into ordered sequences, so that both the elements and their logic relationships are represented.

7.3.5. The need for verbalisation activities

In our approach pupils are involved in problem solving activities, or exploration activities, both in a microworld, or in the paper and pencil environment. We will refer to this activities as to

[36] From now on we are going to talk mainly about this single microworld.

“practical” activities as opposed to verbalisation activities in which the main motive is that of “talking about” practical activities using verbal semiotic systems. Verbalisation activities can be focused also on talking about other verbalisation activities; collective discussions, edition of the *notebook* and writing reports, are different kinds of activities with a common element: verbalisation, i.e. express oneself by word.

In the limits of this thesis, we cannot carry out a detailed analysis of the dynamics between such different activities, in particular, taking into account the use of different kind of semiotic systems (registers [32], Duval,1995). in the following we may refer to all of them using the generic term "verbalisation activity" in order to distinguish them from “practical” activities taking place within the microworld, and within the paper and pencil environment.

The key aspect concerning verbalisation activities when using a microworld as an instrument of semiotic mediation, is that when practical activities are verbalised, pupils enrich their shared semiotic system with new words, gestures, symbols, and drawings, referring to what they experienced in the microworld. Thus new signs are created and shared by the class, with meanings originated in the microworld; starting from these new signs, the teacher can trigger a semiotic process leading to the evolution of meanings originated in the microworld toward mathematical meanings. The signifying forms (words, symbols, gestures, etc.) of these new signs can be used as pivot, referreing both to meanings related to the microworlds, and both to new, derived, mathematical meanings. it may happen that sometimes new hybrid symbols are introduced and function as pivot, this is consistent with a Vygotskian perspective, in fact, where the production of new signs is assumed to play a key role in the production and evolution of meanings, as it permits communication and involvement of new meanings into discourses.

For these reasons, verbalisation activities constitue a key element of the teaching/learning process. In fact, on the one hand they guarantee more expressiveness, on the other hand they facilitate the production of signs to be used and shared in the social discourse, leading to production and evolution of meanings. In the following chapters we will discuss more in details on this point, showing how the teacher can exploit news signs derived from the microworld, as pivots for guiding the evolution of their meanings (referring to the microworld) to mathematical meanings.

Once a practice is verbally expressed, it is possible to talk about it, and once the culture of the class is fixed in a notebook, it is possible to talk about it and eventually to compare it with what is written in the mathematics textbooks.

7.3.6. General strategy to guide the evolution of meanings

According to our hypotheses, the meanings, raising from phenomenological experiences within the microworlds, have to evolve, under the guidance of the teacher, towards the mathematical meanings the teaching/learning activity aims to. In our teaching experiments, the main structure of class activities can be schematised as shown in Figure 14.

Meanings originated in the phenomenological experience are shared within a collective discussion, fixed in the sets of command of L'Algebrista and then reported in the personal *notebook*. Practical activities are verbalised in the forms of written reports and class discussions leading to the production of the class nothebook and update of the commands of the microworld. The notebook and the sets of command of the microworld, are then cyclically revised in order to formulate their logical structure in terms of the logical relationship between the axioms and the theorems of a theory.

Starting from this general idea we may consider the two different cases of axioms and theorems.

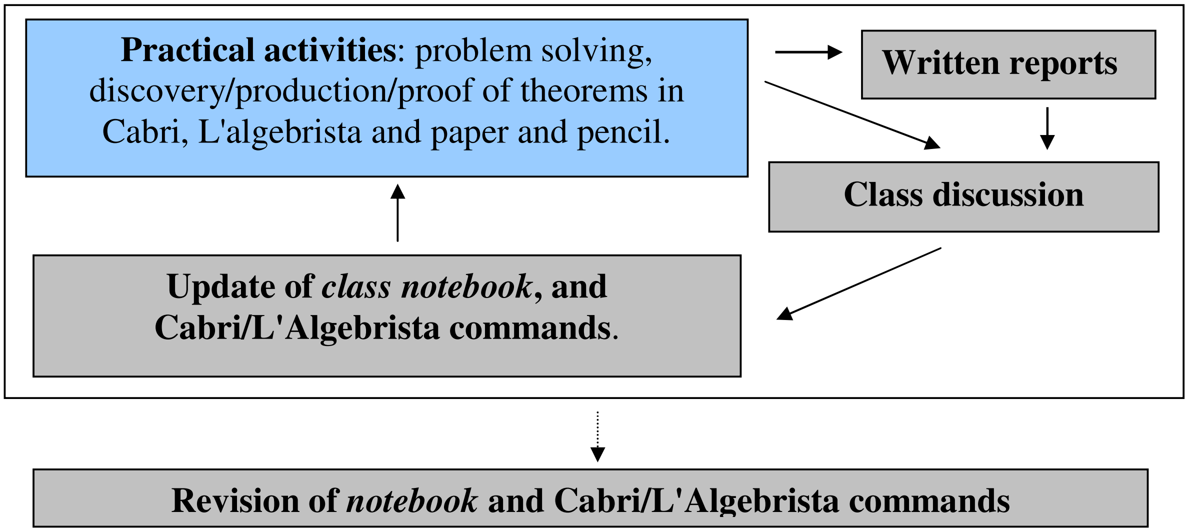


Figure 14 The main structure of the class activities: practical activities are verbalised in the forms of written reports and class discussions leading to the production of the class nothebook and update of the commands of the microworld. The the notebook and sets of command of the microworld are then cyclically revised in order to formulate their logical structure in terms of the logical relationship between the axioms and the theorems of a theory.

*7.3.6.1.**The case of axioms***

One possible way to introduce axioms is to begin working within a microworld: when it is firstly approached by the pupils, it presents a ready made set of commands. Such commands are *given* and can be used to work within the microworld. Thus the pupil *is faced with* a *given* set of commands (to which we may refer also as axiom/command) that *are the only means of action* within the microworld, and that are actually *used to accomplish specific tasks,* that is transform one expression into another one. Such an experience, under the guidance of the teacher, is then verbalised and socialised through a collective discussion, aiming at the formulation and the acceptance of a set of axioms, directly related to the given set of commands. Finally, each axiom is fixed into a statement in the *notebook*. Thus, at the end of this cycle, one obtains a set of commands in the microworld, a set of axioms belonging to the culture of the class, and a set of statements in the notebook; furthermore, the fact that axioms are generated from commands, and statements from axioms, constitutes per se a link between them and may foster the idea that commands, and statements, are both signs representing axioms.

*7.3.6.2.**The case of a theorem***

Once introduced, theorems and axioms can be used to accomplish new tasks, but their status in the culture of the class is different, as the processes generating them. Axioms originate from ready made commands, whilst theorems originate from commands built, by the user, on the commands already available. The dependence relationship, stated between new commands and the commands used to create them provides an operational referent to a logical structure in the organisation of the theory, as it is collectively built by the class. This is certainly a meta theoretical activity, it may be interpreted as a phenomenological experience corresponding to the construction of a mathematical theory, in other words from the experience of the different status of the microworld tools the meaning of theory and that of metatheory may emerge.

When axioms and theorems are reported on the *notebook*, and new commands inserted in the microworlds, they result to be ordered chronologically, however their different status, and the dependence relationships, may not always be evident. For this reason , a specific type of collective discussion

The activity of revising the notebooks gives the opportunity of reflecting and organising the set of axioms and theorems following their logical relationships.

The notebook (personal, but based on shared productions) , the sets of commands of the microworlds and the stated relationship between the two worlds (mathematics and microwolrd), represent the elements on which the history and the "culture" of the class grows and is settled. As a consequence, updating and revising them means to update the class culture and offers the opportunity to develop .

7.4. Summary

In this chapter we presented a paradigm describing the structure of what we called the “cycle of activities” concerning the use of a microworld as an instrument of semiotic mediation. The picture that we sketched focuses the motive of alternating practical activities (within microworld and in the paper and pencil environment) and verbalisation activities (writing reports, class discussions, edition and revision of the *notebook*). The phenomenological experience derived from practical activities, originates meanings rooted in the used microworld. Thanks to verbalisation activities, such meanings are expressed by means of signs whose signifying forms are hybrid, in the sense that they belong to verbal semiotic systems, but are derived from the microworld. Such signs they can be used as pivots to bring their original meanings out of the context of the microworld, guiding their evolution toward mathematical meanings consistent with the teachers’ intentional mathematical knowledge. In this sense, the microworld can be used as an instrument of semiotic mediation.

In the next chapters we will discuss and exemplify the actual functioning of this paradigm. We will show what signs can be derived from L’Algebrista as efficient means for guiding the evolution of meanings, and we will give examples of how such evolution can be guided.

8. The experiment: main lines of the sequence of the proposed activities

In the following we are going to describe the milestones of educational paths that our classes followed during the two long term experiments.

What we present here is the result of a process where class practice was continuously interlaced with research analysis, each of them functioning as input for the evolution of the other (see 6.). Thus, together with the key ideas of the sequence of proposed activities, we will present some educational considerations, explaining how, along with the experiment, some steps of the sequence evolved consequently to the feedback of classroom practice.

8.1. Numerical expressions

The prerequisite that we assume is that pupils can handle computations with integer, fractions, and expressions with integer or rational coefficients. Usually in Italy, in grades 6 to 8, they do a lot of work on the subject, and we assume pupils to have had enough experience related to computation of numerical expressions.

8.1.1. Getting information (on pupils' interpretation of expressions)

At the beginning of the first experiment we submitted a test to the pupils, aiming at verifying if their approaches to numerical expressions were more likely to be procedural or structural. With *procedural* we referred to an approach interpreting expressions only as entities that "have to be computed", while with *structural* we referred to an approach based on comparison of expressions in terms of their structures. The results of the test suggested a prevalence of procedural perspectives among the pupils, thus we chose to start interpreting numerical expressions as computational procedures, pointing toward activities of comparison between expressions; in such a way we wished pupils' perspectives to evolve from procedural to structural.

At the beginning of each of the following experiment we submitted the new pupils to similar tests, with slight changes, always obtaining qualitatively equivalent results, thus we kept the original starting approach mentioned above, that we are going to describe below.

8.1.2. Numerical expressions as computation procedures: introducing equivalence relationships

The first step of the experiment consists of introducing the idea that numerical expressions can be compared in terms of their equivalencies. In order to do so, two different kinds of equivalence relationship are introduced, one based on computing the expressions, the other based on transforming them according to the properties of the operation.

*8.1.2.1.**Equivalence relationship based on the numerical result of expressions***

Def 10: two numerical algebraic expressions are said to be equivalent iff, when computed, they give the same result.

Such a relationship, being based on the equality of numerical results, is transitive, symmetrical, and reflexive, thus it is an equivalence relationship.

Expressions are interpreted as computational procedures which are equivalent if, once executed, they lead to the same number (see also 4.2.2.2). For instance we may consider the two expressions **3•2+3•(5+4)** and **3•(2+5)+3•4** and compare them by executing the computations according to the precedence rules for computing expressions, as shown in Table 2.

	Expression 1	Expression 2
Step 0	**3•2+3•(5+4)**	**3•(2+5)+3•4**
Step 1	**3•2+3•9**	**3•7+3•4**
Step 2	**6+27**	**21+12**
Step 3	**33**	**33**

Table 2 Comparing two expressions: because the final results are the same, Expressions 1 and 2 are equivalent, according to Def 10:; moreover, the expression of each step is equivalent to all the other expressions, because they all lead to the same number once executed as computation procedures.

The two expressions are equivalent, because when computed they lead to the same numerical result, **33**. If we consider the expressions obtained after each computational step, all these expressions lead to the same number, **33**, so they are all equivalent expressions; furthermore, if we consider the number **33** as an expression, then all the considered expressions, beside being equivalent to **3•2+3•(5+4)** and **3•(2+5)+3•4**, also are equivalent to the expression **33**.

The idea of numerical equivalence is introduced together with a principle of equality that can be stated in the following definition as:

Def 11: two numerical algebraic expressions are said to be equal iff, they are identical, i.e. they do not differ at all.

In general, we interpret two objects as being equal when they are exactly the same, from every point of view. In the case of the expressions, two expressions may be equivalent according to Def 10:, but they may not be identical, thus they may be not equal in terms of Def 11:. Two equal expressions are also equivalent, while the converse is not true. For instance, **3+4** is equal to **3+4** but it is not equal to **4+3**, whilst it is equivalent to **4+3** because of the commutative property of the sum.

*8.1.2.2.**Equivalence relationship based on the properties of the operations***

The properties of sum and multiplication on rational numbers, tell us when certain computations procedure give the same results, without executing the procedures themselves (or what ever the number are involved the two procedure will give the same result). In fact given three numbers **N**, **M**, **L**, then:

- The ***commutative property*** of sum and multiplication tells us that the result of **N+M** is the same as that of **M+N**, and that of **NˌM** is the same as that of **MˌN**; thus interpreting the computational procedures as expressions, according to Def 10:, **N+M** is equivalent to **M+N**, and **N•M** is equivalent to **M•N**;

- The ***associative property*** of sum and multiplication tells us that the result of **N+(M+L)** is the same as that of **(M+N)+L**, and that of **Mˌ(NˌL)** is the same as that of **(MˌN)ˌL**; thus interpreting the computational procedures as expressions, according to Def 10:, **M+(N+L)** is equivalent to **(M+N)+L**, and **Mˌ(NˌL)** is equivalent to **(MˌN)ˌL**;

- The ***distributive property*** of multiplication with respect to sum, tells us that the result of **M•(N+L)** is the same as that of **M•N+M•L**; thus interpreting the computation procedures as expressions, according to Def 10:, **M•(N+L)** is equivalent to **M•N+M•L**;

Such properties can be interpreted not only as instruments for checking if two expressions are equivalent, but also (they can be interpreted) as instruments for transforming an expression into an equivalent one. In fact, given an expression, the properties of operations can be used to obtain a new expression, which will be equivalent to the given one. For instance, given the expression **2•(3+4)**, and interpreting **2** as **M**, **3** as **N**, and **4** as **L**, the distributive property gives us a rule to transform it into **2•3+2•4**, corresponding to **M•N+M•L**. Such instruments can be applied not only to entire expressions, but also to sub expressions, in fact given an expression, using Def 10:, it can be easily proved that if we modify one of its sub-expressions, substituting it with an equivalent sub-expression, then the new obtained expression is equivalent to the given one. In other words, starting from an expression, it is possible to build a chain of equivalent expressions, which are all equivalent to each other. Each building step of the chain can be obtained either by means of numerical computations, or by means of the properties of the operations, as described above.

Given the above discussion a new equivalence relationship can be defined:

Def 12: two numerical algebraic expressions are said to be equivalent if, it is possible to transform one into the other, by means of the properties of the operations.

For instance, if we consider again the two expressions **3ˌ2+3ˌ(5+4)** and **3ˌ(2+5)+3ˌ4**, instead of computing their results, we can transform either the first into the second, or vice versa, or the two of them into a third equivalent expression (see Table 3, Table 4, and Table 5).

	Expression 1	Expression 2
Step 0	**3•2+3• (5+4)**	**3• (2+5)+3•4**
Step 1: distributive property	**3• (2+(5+4))**	
Step 2: associative property of sum	**3• ((2+5)+4)**	
Step 3: distributive property	**3• (2+5)+3•4**	

Table 3 Comparing two expressions: Expressions 1 and 2 are equivalent, because it is possible to tranform the first into the second; moreover, the expression of each step is equivalent to all the other expressions, because they all can be transformed one into the other. For each step, we reported the property used, and highlighted the subexpression where it was applied.

	Expression 1	Expression 2
Step 0	**3•2+3• (5+4)**	**3• (2+5)+3•4**
Step 1: distributive property		**3• ((2+5)+4)**
Step 2: associative property of sum		**3• (2+(5+4))**
Step 3: distributive property		**3•2+3• (5+4)**

Table 4 Comparing two expressions: Expressions 1 and 2 are equivalent, because it is possible to tranform the latter into the first; moreover, the expression of each step is equivalent to all the other expressions, because they all can be transformed one into the other. For each step, we reported the property used, and highlighted the subexpression where it was applied.

	Expression 1	Expression 2
Step 0	**3•2+3• (5+4)**	**3• (2+5)+3•4**
Step 1: distributive property	**3• (2+(5+4))**	**3• ((2+5)+4)**
Step 2: associative property of sum		**3• (2+(5+4))**

Table 5 Comparing two expressions: Expressions 1 and 2 are equivalent, because it is possible to tranform the two of them into a third equivalent one; moreover, the expression of each step is equivalent to all the other expressions, because they all can be transformed one into the other. For each step, we reported the property used, and highlighted the subexpression where it was applied.

We observe that, as shown by the example, the properties of the operations, being equivalencies, correspond to transformations that can be inverted: each property, as a tranformation rule, is the inverse of itself. As a consequence, the relationship is reflexive, symmetrical and transitive, thus it is an equivalence relationship.

*8.1.2.3.**Didactical notes***

The idea of comparing expressions, in terms of their equivalence, is introduced through a mathematical discussion orchestrated by the teacher, focused on the elaboration of the meanings of the words "uguale" and "equivalente" (en.: "equal" and "equivalent"). The discussion starts by asking pupils what they mean, in general, with those two words, in reference to their past experience (either schoolastic or common life experience), which constitutes a base for the discussion. The meanings arising from this phase, are then employed to discuss the question of comparing numerical expressions, which is posed explicitly, on purpose, by the teacher. The aim of the discussion is that the class reaches an agreement on the meanings of the words *equivalente* and *uguale* (to which from now on we will refer as *equivalent* and *equal*) in the case of numerical expressions, according to Def 10:, and Def 11:. A side outcome of the discussion is the introduction of the idea of comparing expressions instead of simply computing their result.

The evolution of meanings related to *equivalence* and *equality* of *expressions*, starts from a phenomenological experience constituted by the background of the pupils concerning the computation of numerical expressions, and ends up in the acceptance of the afore mentioned definitions through a mathematical discussion. The result of the discussion is crystallised in the form of definitions reported in the class algebra notebook, according to the principles described in chapter 6. .

Starting from the idea that numerical expressions can be compared, the teacher introduces the well known properties of the operations (Def 12:) from a new perspective: they can be considered as means for establishing the equivalence of expressions avoiding computations. The new equivalence relationship is introduced to pupils through a mathematical discussion continuing the one previously described, leading to the insertion in the notebook of the principle expressed by Def 12:.

The context of numerical expressions as computation procedures that can be compared in terms of equivalence, constitutes an environment where it is possible do develop a field of experience concerning proving statements of equivalence, which is the focus of the following step of the educational sequence.

8.1.3. Proving equivalencies of numerical expressions

For each of the two mentioned definitions of equivalence, we introduce a corresponding practice based on the idea of comparing expressions. The two practices are presented on purpose as identified by two different verbs "dimostrare" and "verificare", to which we will here refer as *proving* and *checking*, and are defined as follow:

Def 13: We say that we *check* the equivalence of two expressions when we compute and compare their numerical results.

Def 14: We say that we *prove* the equivalence of two numerical expressions when we transform one into the other (or both of them into a third expressions), using the properties of the operations.

Given these definitions, it is possible to set up activities of *checking* and *proving* equivalencies (or non equivalencies) between numerical expressions (see appendix for examples, **13.2.**). In particular we observe that, according to Def 14:, *proofs* are carried out by means of the properties of the operations, thus by means of the axioms of a theory.

*8.1.3.1.***Didactical notes**

Of course, from a strictly mathematical point of view, the two practices can both be considered as activities of proving, however the second one (Def 11:) has the peculiarity that it is characterised by an explicit, and goal oriented, use of the properties of the operations as instruments for proving. Furthermore, *proving*, can be interpreted as an algebraic way to handle numerical expressions, thus constituting a possibility of moving from arithmetic to algebra (4.2.2.1), coherently with the approach to algebra that we previously described (see 4.3.).

The first definition (Def 10:) is derived directly from the sphere of practice of computation of expressions, a source of phenomenological experience, in the paper and pencil environment, that in this phase we assume as a prerequisite. We introduce a new practice, which is oriented toward a theoretical perspective of proofs within a theory, and which is presented as distinguished from the sphere of practice of numerical computations. The dinstinction between the two practices constitutes a source for setting up *mathematical discussions*, as *polyphonies of articulated voices* (see 0) on the issue of comparison of expressions: pupils' experience in numerical computations constitutes a source for the voice of a consolidated practice, whilst pupils' new experiences in *proving* equivalencies, constitutes a source for the voice of theory.

At this point, because we aim at an evolution of the idea of proving, toward the mathematical idea of proving within a theory, according to our hypotesis we need a sphere of practice providing pupils with adeguate phenomenological experience, to be exploited by the teacher to orchestrate mathematical discussions. Consequently, L'Algebrista is introduced in classroom practices, as an environment where it is possible both to check and to prove equivalencies between expressions. The two kinds of practices, are kept separated by the interface of the software, which presents buttons corresponding to the properties of the operations, as dinstinctly separated by buttons that execute numerical computations (see Figure 15).

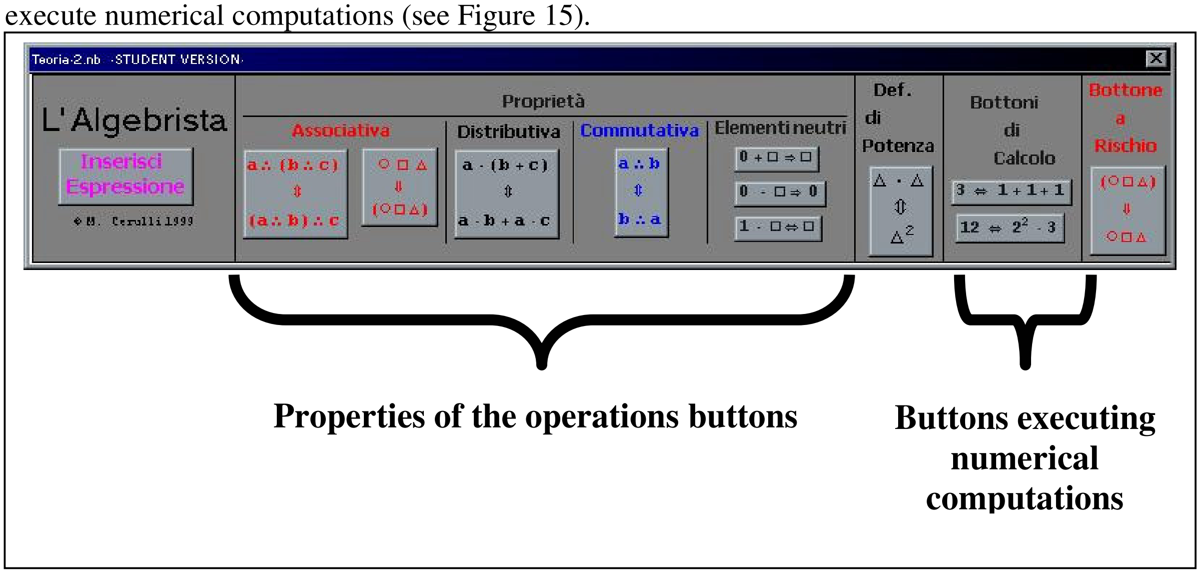


Figure 15 Teoria2, this contains buttons corresponding to axioms such as the basic properties of the operation; it contains a button incorporating the definition of power; it contains buttons for numerical computations.

Ativities of various kinds are proposed to pupils, focusing on the central idea of comparing expressions and proving equivalencies[37], to be accomplished with *L'Algebrista*. In this way, because of the nature of the microworld, pupils engage in activities where the properties of the

[37] This includes activities requiring checking and proving equivalencies, and activities presenting pupils with given proofs whose sequences of used commands are hidden, and requiring them to individuate the original sequence of commands (see 13.2.)

operations, thanks to their computational counterparts 6.2. , acquire the status of instruments, by means of which it is possible to transform one expression into the other, thus they become a means to consruct a proof of the equivalence betwee two expressions. In parallel, similar activities are proposed within the paper and pencil environment, so that we have two spheres of practices concerning proving, which constitute sources for different voices to be exploited by the teacher to orchestrate mathematical discussions[38].

The natures of the phenomenological experiences related to the two environments are very different: in the case of paper and pencil the pupil is in charge of producing himself/herslef the effects of applying a property of the operations to a given expressions; whilst in L'Algebrista such effect is produced by the computer, so the pupils can experience its effects phenomenologically, in the sense that it is a phenomenon which is independent external to the pupil. Buttons in L'Algebrista are real instruments for transforming screen expressions, thus they can be the source of the instrumental aspect of the meaning of the operations properties; on the other hand, in the paper and pencil environment, there is no instrument, external to the pupil, corresponding to the properties of the operations. Again, such difference can be exploited as a source for voices to be orchestrated by the teacher in mathematical discussions: the voice of L'Algebrista, and the voices of the single pupils. Moreover, in paper and pencil, a pupil may transformation an expression into another one without being conscious of the equivalence principle he/she is using. It may happen that pupils learn how to transform an expression, but they lose very soon, if they ever had it, the counsciousness of the equivalence principles underlying the performed transformations. On the contrary in L'Algebrista, in order to transform an expression, one can't avoid referring to the underlying equivalence principles that are identified by the buttons of the microworld. In fact, it is not possible to transform an expression without clicking on a button. The button functions as an external control which determines pupils taking counsciouness of the equivalence relationship underlying the transformation they are performing.

What pupils experience phenomenologically is expressed in written individual reports, and later within mathematical discussions, which can be orchestrated on the basis of different voices coming alternatively from the sphere of practice and that of theory. In fact, thanks to the introduction of L'Algebrista, and thanks to the dinstinction between *proving* and *checking*, the pupils are given several kinds of practices, and corresponding voices to be used in mathematical discussions: checking and proving in L'Algebrista, and checking and proving in paper and pencil. For what concerns possible voices of the theory, these can be represented either by the class algebra notebook, by the teacher, and by L'Algebrista, which, because of the mathematical knowledge it embeds, can be taken as representative of mathematicians, just as the teacher is.

Through the parallel development of activities within L'Algebrista, and the paper and pencil environment, and through verbalisation activities, we begin to build a relationship between the two environments, either in terms of analogies, either in terms of differencies. Such relationship, as we already discussed, is central if we want pupils to correctly interpret L'Algebrista as a mathematical instrument (see 6.3.), and is fostered also through verbalisation activities such as class discussions, and edition of individual reports.

*8.1.3.2.**Research notes***

During the pilot study (7.1.), we dedicated a few weeks to the phase of numerical expressions, because we assumed that, before being able to master literal expressions, pupils needed a certain level of mastery of numerical expressions in the proposed algebraic way. The data collected suggested to us that this level of mastery did not need to be very high, and we also observed that a long time spent on numerical expressions resulted in a decrease of pupils' interest and involvement

[38] For instance asking pupils questions such as "what would L'Algebrista do here?", "Would L'Algebrista accept this?", etc.

in the proposed activities. Consequently, in the following experiments the time dedicated to numerical expressions was reduced gradually to only a few lessons.

8.2. Literal expressions

Once the practice of proving equivalencies of numerical expressions is somehow consolidated, we can approach literal expressions, on the basis of a practice which is algebraic, and does not need to be radically changed in order to be extended to the case of literal expressions. In order to do so we need to introduce an equivalence relationship between literal expressions.

8.2.1. Equivalence of literal expressions

On the basis of the equivalence relationships defined for numerical expressions, two new definitions are proposed for the case of literal expressions.

Def 15: *Equivalence by means of computations*: two algebraic expressions are said to be equivalent if, for any number that we substitute for letters[39], we obtain two equivalent numerical expressions.

Def 16: *Equivalence by means of the properties of the operations*: two algebraic expressions are said to be equivalent iff, it is possible to transform one into the other (or both into a third one) by means of the properties of the operations.

Thus, as in the case of numerical expressions, we can compare literal expressions, and eventually prove their equivalence or non equivalence.

If we interpret the properties of the operation as the axioms of a theory, we can reformulate the latter as:

Def 17: *Equivalence by means of the axioms*: two algebraic expressions are said to be equivalent iff, it is possible to transform one into the other (or both into a third one) by means of the axioms of a theory[40].

*8.2.1.1.**Didactical notes***

At this point of their scholastic history, even if we don't assume pupils to be familiar with literal expressions, these mathematical objects are not completely new to them. In fact they have been facing literal expression at least in two forms: in the forms of formulas for computing areas of geometrical figures; in the form of formulas for transforming expressions, which is the case of the properties of the operations, and in the inscriptions on the corresponding buttons of *L'Algebrista* (see 6.1.2.3). As a consequence, to introduce symbolic manipulation of literal expressions, we chose a type of activity which stands between the two practices: we present pupils a set of formulas representing the area of a geometric figure and ask them to chose which of the given formulas are correct (see 13.3.). This involves either an interpretation of the formulas as computation procedure, or a need for comparing the formulas, because if two expressions are both correct, then they must give the same numerical results once used as formulas for computing areas. Drawing on the outcomes of this introductory activity, the teacher orchestrates verbalisation activities leading the class to share the new definitions of equivalencies, which are then reported in the *mathematical notebook*.

The key idea of this phase is to situate literal expressions in the context of two consolidated practices that correspond to the two principles of equivalence that we defined, thus offering a sort of

[39] Of course a chosen number has to be substituted to every occurrence of the chosen letter in both the expressions.

[40] In general here we refer to the axioms of rings of polynomials with rational cohefficients, in fact, what is usually called "expression" in school mathematics, can be interpreted as a polynomial. With that axioms, thanks to the foundamental theorem of algebra, it is possible to prove that the three definitions are equivalent.

"ready made" sphere of practice for comparison of literal expressions. In other words, instead of introducing algebraic practices and algebraic objects at the same time, we firstly introduce algebraic practices on more familiar arithmetical objects, the numerical expressions, and once such practices are consolidated, we enrich the practices with new algebraic objects, i.e. literal expressions. The practices concerning numerical expressions can evolve and include literal expressions, thus constituting a new sphere of practice, that of symbolic manipulation.

*8.2.1.2.**Research notes***

Within the proposed introductory activity (see 13.3.), pupils are not only required to conjecture which of the proposed formulas are correct, but they are also asked to justify their answers. They are left free to use both *L'Algebrista* and paper and pencil, and are left free to refer, if they want, to any of the definitions of equivalence on numerical expressions, which constitute their background knowledge.

Since we approached this phase for the first time during the pilot study, we observed an interesting behaviour amongst the pupils. During the phase of production of their conjecture pupils proceeded in a variety of ways, all leading them to the individuation of at least one specific correct formula. On the contrary, in the justifying phase, all of them proceeded in the same way: they individuated a correct formula as a referent, and *proved* the correctness of other formulas by transforming them into the first one by means of the properties of the operations (some did it in paper and pencil, others with *L'Algebrista*). This behaviour suggested to us that this kind of activity, situated in that particular moment of the educational sequence, resulted to be an effective junction node between numerical and literal expressions. As a consequence it became one of they key milestones also in the sequence followed in the long term experiment.

8.2.2. Proving equivalencies of algebraic expressions

As far as literal expressions are concerned, the considered definitions of equivalence (see 8.2.1) play different roles at an operative level, the first one is optimal for proving that two expressions are not equivalent, while the latter ones are optimal for proving that two expressions are equivalent.

For instance we shall consider the example of the expressions **A•B+A•(C+D)** and **A•(B+C)+A•D**, we can substitute numbers for the letters, for instance we can substitute **A** with **3**, **B** with **2**, **C** with **5**, and **D** with **4**, thus obtaining the two numerical expressions that we compared in section 8.1.2.1 and that we proved to be equivalent. However, if we want to prove, by means of computations (Def 15:), that the given literal expressions are equivalent, then we will need to perform infinite substitutions of numbers to the letters, thus in practice this definition cannot be used to prove equivalencies. In that case, instead, it is possible to use the equivalence by mean of the properties of axioms, as shown in the example of Table 6.

	Expression 1	Expression 2
Step 0	**A•B+A•(C+D)**	**A•(B+C)+A•D**
Step 1: distributive property	**A•(B+(C+D))**	
Step 2: associative property of sum	**A•((B+C)+D)**	
Step 3: distributive property	**A•(B+C)+A•D**	

Table 6 Expressions 1 and 2 are equivalent, because it is possible to transform the first into the second by means of the properties of the operations. For each step, we reported the property used, and highlighted the sub expression where it was applied.

On the contrary, if one needs to prove that two expressions are not equivalent, the definition by means of axioms is not effective; in fact, if one cannot transform an expression into another, it may be the case that he/she simply doesn't see how to do it, but it does not mean that it is impossible to do it. In this case it is better to use the definition based on numerical computation, in fact, in order to prove that the equivalence does not hold, it is enough to show a counter example; it is enough to find a set of number to substitute to the letters which originate a couple of non equivalent numerical expressions. From a mathematical point of view it is correct to move from one definition to the other because they result to be equivalent.

*8.2.2.1.**Didactical notes***

Within this phase pupils are set activities in which they are required to compare a set of algebraic expressions, to conjecture and to prove which of them are equivalent or not equivalent (see 13.4.). The aim is to consolidate the practice of comparing algebraic expressions alternating *L'Algebrista*, and the paper and pencil environment. The related phenomenological experiences will constitute the basis for the evolution of meanings related to proofs of equivalencies toward the idea of proving theorems within a theory.

*8.2.2.2.**Research notes***

Here we observe that the distinction between *checking* and *proving* previously introduced (see 8.1.3) acquires a clearer practical meaning because of the different roles played, in expressions comparing activities, by numerical computations and by axiomatic transformations; in fact to substitute numbers for the letters turns out to be an optimal method for producing conjecture on the possible equivalence of two given expressions. In other words, substituting numbers becomes the most immediate way to actually check if two expressions can be equivalent or not. In case the two numerical expressions give different results, then they can be interpreted as a counter example showing and proving the non equivalence of the two expressions. On the contrary, in case the results are identical, one can conjectures that the two literal expressions are equivalent, the conjecture can be proved by means of the axioms.

8.2.3. Proving theorems and building a theory

Every transformation of an expression into another one by means of the axioms can be interpreted as a proof of their equivalence. Among all the possible equivalencies between literal expressions, there are some which in mathematics are considered as more important than others, and play a relevant role in standard symbolic manipulation. In fact, the transformation rules used in standard symbolic manipulation rely on the equivalences of a particular couple of expressions, that can be interpreted as theorems proved by means of the axioms. For instance, in our experiment we present as theorems the following principles for manipulating algebraic expressions:

- Sum of monomials;
- Rules for managing powers;
- Rules for managing fractions;
- "Prodotti notevoli", i.e. standard formulas such as that of the square of the binomial and others.

These theorems, together with the axioms corresponding to the properties of the operations, constitute an algebraic theory, the theory of standard symbolic manipulation, i.e. the theory within which any standard manipulation constitutes an equivalence theorem.

*8.2.3.1.**Didactical notes***

Any equivalence is firstly derived through a transformation activity (could it be related to L'Algebrista or to paper and pencil), if it is officially selected as one which deserves to be saved, it will be inserted both in the *class algebra notebook* and in L'Algebrista (in the form of a button). Both the decision to select an equivalence and that of inserting the corresponding theorem in the *class algebra notebook* and in the microworld are the result of a collective decision reached by means of mathematical discussions.

*8.2.3.2.**Research notes***

In this phase the role of L'Algebrista, as counterpart of paper and pencil, is crucial because of its feature of not executing any implicit automatic transformation. In fact, this feature of the microworld results to be a motivation for pupils to engage in transformations that they would skip in a paper and pencil environment.

Often symbolic manipulation is presented as a set of arbitrary rules, one independent from the other, that must be memorized resulting in an activity that pupils may find difficult to control. On the contrary L'Algebrista provides pupils with a context in which symbolic manipulation is introduced as the practice of transforming expressions starting from a limited and well identified set of shared principles, the properties of the operations represented by corresponding buttons. The transformations that is possible to perform in L'Algebrista are all those, and only those, that can be realized using the available buttons, which guarantee the correctness of the obtained equivalences. Thus, also if one is working in paper and pencil, if he/she is not sure of the validity a performed transformation, he/she can still refer to L'Algebrista to find an answer to his/her doubts. In this sense, the use of symbolic calculation rules acquire a theoretical meaning that otherwise it would be difficult to be grasped.

L'Algebrista allows the teacher to introduce the control also on transformation rules that are already known to pupils and well automat zed. referring to L'Algebrista offer the opportunity of make explicit certain transformation steps on which otherwise it would be difficult to direct pupils attention. For instance if pupils have to transform the expression **A+(-1)•A** into **0**, in paper and pencil they simply cancel **A+(-1)•A** and substitute it with **0**; on the other hand *L'Algebrista* itself does not compute automatically the sums of monomials, i.e. there is no button which executes this step automatically, thus the expression **A+(-1)•A** can be transformed into **0** only by means of the distributive property and other axioms/buttons. Once the transformation steps are made explicit, it is possible to interpret them as a proof that **A+(-1)•A** is equivalent to **0**. Because this statement appears, also to pupils, to be a useful one, then it is a good candidate for becoming a theorem of the class mathematical theory, and thus it is inserted in the notebook and a button is made with the Teorematore and added to *L'Algebrista*. We cited here the example of this theorem because it was the first one our pupils proved in the study pilot, and it was on the feedback of that experience that we decided to add to *L'Algebrista* the special feature of enabling users to add buttons, constituted by the Teorematore. In fact, the software itself evolved together with the experiment, according to the feedback, and according to pupils suggestions.

8.2.4. Factorisation

Factorising expressions is one of the main issues of the curriculum for grade 9 in Italian school. given an expression, it consists of the activity of finding another equivalent expression which is structured as the product of other expressions. Factorised forms are very important, for instance in the solution of problems involving finding the zeros of polynomials.

*8.2.4.1.**Didactical notes***

The sphere of practice of comparing expressions and proving equivalences, provides pupils with substantial experience involving either goal oriented manipulations of expressions, either substitutions of numbers to the letters. In the first part of the sequence, symbolic manipulation is devoted to transform expressions, to prove equivalence, in the second part pupils are presented with a new kind of problem. Given two expression, one may question what numbers can be substituted to the letters in order to originate numerical expressions which, once computed, lead to the same result.

In order to solve this kind of problem, previous experience of transformation becomes fundamental, providing a way of reformulating the problem in terms of transforming an expression into an equivalent one suitable for solving the problem.

Such new kind of problem introduce a new practice, that of goal oriented symbolic manipulations, not aiming at proving equivalencies of expressions, rather to find solution of equations.

8.3. Equations

The last step of our educational sequence is that of introducing the issue of solving equations. This issue is introduced as a natural evolution of the practice of comparing expressions.

8.3.1. Introducing equations

We interpret equations as a an open question: "given two literal expressions, what are the numbers that, substituted for the letters, originate equivalent numerical expressions?". This question can be posed for any couple of expressions, even equivalent ones, in fact in that case, the set of solutions corresponds to the whole set of numbers within which they can be searched, which in our case has infinite cardinality. However, the most interesting case is that of non equivalent expressions, in that case the problem is to individuate set of solutions, if there are any.

In algebra there are techniques for solving equations that are based on two main kinds of transformations; given the equations **A=B** (where **A** and **B** are algebraic expressions) it is possible:

- To transform **A** and/or **B** into expressions **A'** and **B'** which are respectively equivalent to **A** and **B**;

- To transform **A** and/or **B** into expressions **C** and **D** which are not equivalent to **A** and **B**, but the new obtained equation **C=B** is equivalent to the equation **A=B** in the sense that it has the same set of solutions;

The first type of transformations corresponds to the set of transformations of expressions, discussed in the previous sections, those which pupils are familiar with.

The second type of transformations corresponds to the standard algebraic techniques for solving equations , which consists in transforming them into a new equation whose form makes it easier to see what are its possible solutions. For instance, given the equation $\mathbf{x^2\text{-}x=0}$ one can transform it into **x(x-1)=0** and then, using the properties of the neutral element of the sum, it is possible to deduce that the numbers **0** and **1** are solutions of the equation. Or, given the equation **x-2=3** one can transform it into **x=5** and then deduce that the number **5** is a solution of the equation.

*8.3.1.1.**Didactical notes***

As said above, the issue of equations is presented in a problematic way, starting from the background of the practice of comparing expressions. The main question about solutions (i.e. the numbers that, substituted into the two expressions, originate two equivalent numerical expressions), is presented by means of mathematical discussions, leading to the formulation of two basic techniques for helping in solving this problem, which is named as the problem of solving equations. The principles individuated are the following:

- **A=B <=> A-B=0**
- **A=B (with B=/=0) <=> A/B=1.**

Mathematical discussions leads to an agreement of accepting these principles as axioms to be used for solving equations, thus they are reported on the *mathematical notebook* and two corresponding axioms/buttons are introduced in *L'Algebrista* (by means of the Teorematore). Consequently other standard principles for solving equations are proved by our pupils by means of these two principles and by means of the theory of equivalent expressions they previously developed. In other words, rules such as that of cancelling elements of the two terms of equations, or that of adding or multiplying the two terms for a given expressions, are not presented as ready made principles. On the contrary, they are produced by the class as theorems, following the same educational approach used for theorems concerning manipulation of simple expressions. Each of the produced rules is inserted in the notebook and in *L'Algebrista* in the form of a theorem/button.

This phase of introduction to equations represents the final step of our approach to algebra with *L'Algebrista*, in fact, once introduced, the principles for solving equations, the software turns out not always to be very effective as a practical instrument for proving equivalencies of equations. This leads our pupils to feel the need to get rid of an instrument which at this point results in being unable to do things that they are able to do. From this point, the teacher begins to refer to *L'Algebrista* less frequently in order to favour pupils getting rid of *L'Algebrista*; the aim is to point toward the direction of a mathematical knowledge freed from references to *L'Algebrista*, a knowledge whose class counterpart is represented in the *mathematical notebook* of the class.

*8.3.1.2.**Research notes***

In the pilot study we *did not* reach the phase of equation, the experiment ended after the first activities of theorem proving within the domain of algebraic expressions. In the following long term experiment we went further, and the idea of approaching equations came directly from the experimenting teacher, who introduced, by her own initiative, pupils to the problem of equation solving in terms of comparison of expressions, as we described above.

Consequently, following the teacher's initiative, and at the requests and suggestions of the pupils, we developed a new feature of *L'Algebrista*, that of the *speaking buttons* (see 6.1.2.4) which previously were not implemented. This feature was essential for buttons requiring the user to input the expression to be multiplied with, or summed to, the two terms of a given equations.

Finally, the feedback we got from the equation phase, gave us clear indications as to what can be a good moment for quitting *L'Algebrista* along the development of students scholastic history.

9. Exploiting the relationship between two fields of experience as a mean for guiding the evolution of meanings

In the previous chapters we described our approach to algebra as a theory and our educational perspective based on the idea of using a software, L'Algebrista, as an instrument of semiotic mediation.

The educational goal identified was that of introducing pupils to symbolic manipulation, and to the idea of theory. Within this chapter we will describe/ highlight some elements characterising this twofold objective and show how they evolved, along the experimentation, in the protocols produced by our pupils. The analysis we are presenting is aimed at showing the relationship between pupils (learning) achievements and L'Algebrista used as an instrument of semiotic mediation. We will end up showing an example of how the microworld can be used within our theoretical framework in order to guide the evolution of the meanings that are relevant to our educational goals.

9.1. Preliminary notes

The experiment was carried out in (involved) five different classes (we will call them class 1998, class 1999, class 2000, class 2002, and class 2003, according to the year when each of the groups started the teaching experiment). Each experiment lasted several months, aiming at covering most of the Italian algebra curriculum for grade 9, and part of the curricula of grade 10, for schools with scientific orientation ("Liceo Scientifico"). The aim of this chapter is not to give a comprehensive description of all the data collected along with our research, but it is a qualitative one, and we aim at analysing only some key issues concerning our educational goals and our research objectives. Thus we are going to use protocols picked from the data collected in all the five experiments, specifying in each case which class the data is taken from.

9.2. Symbolic manipulation

In our approach we interpret symbolic manipulation as an activity of goal oriented transformations of expressions, such kind of activities are viewed as means for solving problems. Such view of symbolic manipulation, and the ability to exploit it in problem solving activities, is one of our main educational goals. However, as a propedeutic sub objective, according to the indications found in literature review, we chose that of reaching a structural interpretation of algebraic expressions, as opposed although complementary to the procedural interpretation typical of arithmetic (see **4.2.1**). Within an procedural view, algebraic expressions are viewed simply as processes of computation, whilst within a structural view they are viewed as objects and as such it is possible to involve them in processes that can leading to the production of new objects and related meanings (**[73]**, Sfard, 1991; **[74]**, Sfard et al. 1994). For what concerns symbolic manipulation, the mathematical objects that we will take into account are expressions (both numerical and literal) and symbols of equivalence, while the processes we will focus on are that of numerical computation of expressions, and that of expression transforming. We will show how the experimentation led pupils to a structural view of expressions, and to interpret symbolic manipulation in terms of activities of theorems proving within a theory. In our analysis special attention will be put on the role played by a microworld, L'Algebrista. We will proceed by describing the starting situation, and then will discuss some key steps of the evolution of pupils approach to the subject.

9.2.1. Pupils' procedural view of numerical expressions

Pupils, at the beginning of our experimentation, according to their scholastic background, were familiar basically only with numerical expressions, and interpreted them procedurally, as our preliminary tests confirmed. In fact, expressions were interpreted as something which must be computed following predetermined procedures, and an equal sign ("=") put on the right of an expression was interpreted as an input for starting computation. Moreover, although pupils had some notions concerning the properties of the operations, such properties were interpreted basically as computational rules. Such an procedural view led pupils to produce statements as the following paradigmatic ones (the text of the whole test can be found in appendix *13.1.*).

Question	Pupil's answers[41]
T2. *Write what you know concerning each of the following words and phrases, for instance you can write phrases containing them, or you can explain their meaning. You can also write examples.* *[…]* *13. Expression* *(see T 3 in appendix 13.1. for the complete text of the exercise)*	17. "Expression = set of calculations" 18. "Expression = set of operations with or without brackets that have their rules" 19. "Expression = a series of operations connected on to the other" 20. "Set of mathematical calculations" 21. "Set of operations" 22. "I developed *(ita.: "sviluppato")* a very long expression: Ex. 5+3-4,(6+5)=+8-44=36" 23. "A sequence of operations. Ex 2• (5+4)" 24. "Set of operations, attached on to the other" 25. "A big operation with a certain number of numbers, letters, sums, divisions. Within this mega-operation you must respect the order of the brackets" 26. "A set of operations".
T 1. *Observe the following writings, for each of them explain why you think it is correct, or why you think it is wrong.* • […] • 15 + 6 · 4 + 19 · 4 + 11 = 15 + (6 + 19) · […] • 8 + 9 · (3 + 2) - 17 = 8 + 27 + 18 - 17 • […] *(seeT1 in appendix 13.1. for the complete text of the exercise)*	1."**15 + 6•4 + 19•4 + 11 = 15 + (6 + 19)•4 + 11** it is wrong; **8 + 9•(3+2) – 17 = 8 + 27 + 18 – 17** it is correct because of the distributive property" 2."**8 + 9•(3+2) – 17 = 8 + 27 + 18 – 17** it is wrong because firstly one must compute the brackets" 3."**15 + 6•4 + 19•4 + 11 = 15 + (6 + 19)•4 + 11** it is wrong because, I think that the procedure is the following one: **15+24+76+11=126**";

Protocol 1 We report on the left the text of the preliminary test submitted to pupils; on the right we report some of the answers given by pupils of Class 1998, grade 9. For each question, each number, in the answers' column, stands for the answers given by a single pupil.

[41] We copied and translated pupils' answers keeping the formatting of their answers, and using the same symbols using by them. In other words, also the symbols "=" is used by the pupils exactly how it is reported in this table.

pupils' answers give us an idea about pupils approach to numerical expressions, which turns out to be rather procedural. In fact expressions are seen as set or sequence of operations that have to be executed according to certain given rules that cannot be changed and in particular cannot be inverted, leading to a numerical result. At the same time, the symbol "=" seems to be interpreted as representing asymmetric equivalence relationship, and it has a strong directionality from left to right pointing from an expression toward its numerical result. Finally the rules of precedence are interpreted only in terms of the order to be followed to execute operations, and they doesn't seem to be interpreted in terms of the structure of the expression, which is seen more as a process to be executed, then an object with its structure.

Along with the experimentation pupils' view of expressions showed to evolve, leading to more structural behaviours. We will bring some evidence of such evolution, also discussing how L'Algebrista and the followed educational approach influenced such evolution.

9.2.2. Splitting the word "calcolo" (computation) into the words "verifica" (check) and "dimostrazione" (proof)

In Italian language, in school mathematics, the word most commonly used for symbolic manipulation, is the word "calcolo" (en.: "computation activities" or "symbolic manipulation"), which is polysemic in the sense that it refers both to numerical computations and to transformations of expressions based on transformation rules derived from the properties of the operations. The first interpretation of the word is consistent with an procedural view of expressions, as that showed by pupils at the beginning of the experiment. The second interpretation is consistent to the educational aim of our experiment, according to which, symbolic manipulation is also an activity of transforming expressions by means of axioms. Our aim was thus of associating to the word "calcolo" both the possible interpretations. Starting from the fact that pupils were mainly oriented toward the first interpretation as "computation activities", we split on purpose the word "calcolo" into two different words, "verifica" and "dimostrazione", and exploited L'Algebrista as an instrument of semiotic mediation in order to give relevant meanings to these words. Below we are going to discuss further more the polysemy of the word "calcolo" in order to explain more in details our choice of splitting it into two words corresponding to two different practices.

The word *Calcolo*, among other meanings, is used to refer to algebraic computations, including both, numerical computations, and rule based computations. *Calcolo*, in general, refers to both, arithmetical and algebraic ways for handling expressions. However, such a word is associated mainly to the activity of "calcolare espressioni" which in the case of numerical expressions means "to compute the numerical result" whilst in the case of literal expressions means "to expand and simplify the expression". In both cases one can proceed either computing the results of operations between numbers (when possible), or by using computational rules derived from the properties of the operations. In standard Italian school approaches, pupils experience a lot of *Calcolo* with numerical expressions, then they are presented literal expressions, and asked to "*calcolare*" them, that is, the *calcolo* of numerical expressions is extended to literal expressions, coherently to the interpretation of algebra as generalized arithmetic. This can result in pupils' difficulties when the differences between the numerical case and the literal case are not enough highlighted and elaborated, which is often the case when algebra is presented as generalized arithmetic. In section 4.2.1 we described the difficulties of Francesca, a 10th grade pupil that we interviewed, she could not see the supposed continuity between numerical and literal expressions, simply because with numbers you can compute a result, while with letters you cant, you get stuck, to use Francesca's words: *"[…] Because if I am given 10+3 whilst…[if you give me] a•b+c+d I get stuck… (laughs) I can't work it out."*. The pupil, during the interview explained us that her teacher kept telling her that computing with letters is the same of computing with numbers, but she couldn't see why. We hypothesise a misunderstanding due to the polysemy of the word *calcolo* how it is used in Italian schools, we argue that for the teacher the word *calcolo*, for numerical expressions, referred either to numerical computations, and to rules based transformations. As a consequence the supposed continuity was based on rules based transformations. On the other hand, for Francesca the word *calcolo* probably referred simply to numerical computations, thus she couldn't see the supposed continuity. The example showed a lack of clarity concerning the meanings associated to the word *calcolo*. moreover, our preliminary tests suggested us that pupils could not always manage both the arithmetical and algebraic meanings condensed in the word *calcolo*, even if they were able to compute or simplify expressions.

The key idea of our approach is situated exactly at this point, in fact, if we want to introduce symbolic manipulation within a theoretical perspective, the main focus of our activities has to be on transformations of expressions by means of the axioms of a theory. In other words we propose to interpret the rules based transformations of the *calcolo* as transformations based on the properties of the operations, which we take as the axioms of our algebraic theory. If in previous pupils experience the most relevant meanings of the word *calcolo* were the arithmetical ones, with the introduction of literal expressions, we wanted to introduce the algebraic meaning of *calcolo*, fostering its evolution toward a theoretical view of symbolic manipulation. As a consequence we wanted to distinguish clearly the two different meanings associated to the word *calcolo* in order to avoid confusion and foster the evolution of both meanings within a theoretical perspective.

The first step of our intervention was thus to introduce, through a class mathematical discussion, the idea of comparing numerical expressions questioning their equivalence relationships. Numerical expressions could then be compared either by means of numerical computations either by means of transformations based on the properties of the operations, which the teacher, on purpose, began to call also "axioms". This kind of activity of transformation becomes the core of the activities proposed to pupils, and substitutes the activity of "calcolare" ("compute numerical results" or "simplify") with which pupils were familiar prior to begin the experimentation. The word *calcolo* and the activities of *calcolare* are on purpose eliminated, at the beginning of the experiment, from class practices. As we said, we wanted to split algebraic handling of expressions from arithmetical handling of expressions, as a consequence, given the mathematical problem of comparing expressions, we introduced two new words: *verificare* and *dimostrare*, which in this thesis we translated with "*to check*" and "*to prove*". The meanings of the new words, as they had been introduced in class practices, are strictly tied to the idea of comparing expressions in terms of their equivalence relationship. In fact two expressions, in our experiment, are considered to be equivalent if either their numerical results are the same, or if it is possible to transform one into the other using the axioms of the chosen theory. Once these definitions are discussed and shared by the class, the teacher introduces the new words *to check* and *to prove* (ita.: "*verificare*" and "*dimostrare*") as referring to the two ways to define equivalencies of expressions. *To check* that to expressions are equivalent means to compute their numerical results and to check if the obtained numbers are the same, whilst *to prove* that two expressions are equivalent, means to transform one into the other by means of the given set of axioms[42].

The new introduced words structure a distinction between the arithmetical and the algebraic meanings of the word *calcolo* which is now split respectively in *check* and *prove* (or *proof*). This distinction is forced, on purpose by the teacher, and it is to be considered as temporary. In fact, the meanings fostered through this distinction between words and the relative distinction between activities, are meant to be merged again in the word *calcolo* once the experimentation is over. Our hypothesis is that of creating solid algebraic meaning, as opposed to arithmetical meaning, and that pupils internalised them as such; after that they can merge such meanings in the polysemic word *calcolo* and be able to manage its different meanings.

[42] Of course, the chosen axioms are discussed in class, we generally begin considering only the properties of the operations, and then add gradually other needed axioms to the theory, as we explained in chapter 8.

In the following we are going to show the role played by L'Algebrista in the creation of meanings for the words *check* and *prove*.

9.2.3. The structure of an expression: the role of the selecting tool of L'Algebrista

As observed in section (6.2.1), L'Algebrista expressions, differently from paper and pencil expressions, embed their mathematical tree structure evoking it by means of the modalities of interaction with the software. In fact in this microworld, in order to transform an expression, the first thing to do is to select a sub expression of the expression, and then it is possible to apply a command on the selected part, by simply clicking on the corresponding button. The act of selecting, is unavoidable, and at the same time L'Algebrista doesn't allow the user to select subparts which are not sub expressions from an algebraic point of view. For these reasons we hypothesised that the use of the selection tool could be used as an instrument of semiotic mediation in relation to the tree-alike nature of the structures of algebraic expressions, contributing to the evolution of a structural perspective. The experimentation confirmed our expectations indicating how the selection tool could be exploited as an instrument of semiotic mediation, as we are going to clarify in what follows, by analysing some paradigmatic examples.

9.2.3.1. ***Importing the selection tool into the pencil and paper environment***

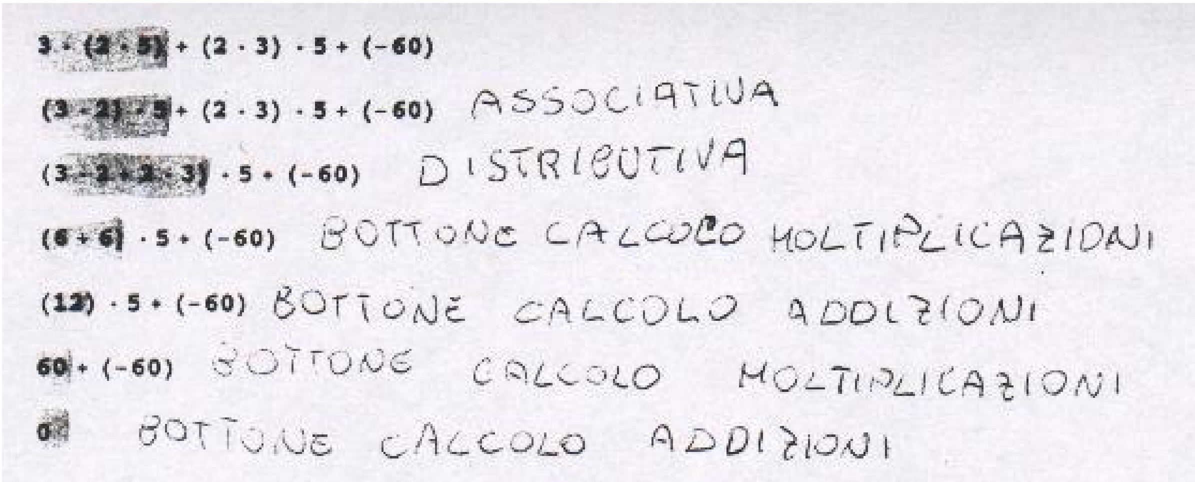
The first example we consider is situated at the very beginning of the experimentation; just after pupils had been working with L'Algebrista for the first time, in their Laboratory, pupils are presented, as homework, two different activities:

- on one hand, pupils were presented two chains of transformations produced within L'Algebrista; for each transformation step the information on the use of buttons and the selected sub expressions was hidden, and pupils were required to indicate, what button was used and what sub expression it was applied on (see activity CS 1 in appendix 13.2.).

- on the other hand, pupils were asked *to check* (by means of numerical computations) and *prove* (by means of axioms based transformations) equivalencies of given expressions in the paper and pencil environment (see activity CS 2 in appendix 13.2.);

The two tasks are to be accomplished without using L'Algebrista, which actually they do not have at home; so at the moment they produced the protocols we are analysing, the microworld was not available at all to them.

For what concerns the first of the two activities, all the pupils behaved in a similar way, by somehow imitating L'Algebrista, as shown by the protocol produced by Nicola in Protocol 2.



$3 \cdot (2 \cdot 5) + (2 \cdot 3) \cdot 5 + (-60)$

$(3 \cdot 2) \cdot 5 + (2 \cdot 3) \cdot 5 + (-60)$ ASSOCIATIVA

$(3 \cdot 2 + 2 \cdot 3) \cdot 5 + (-60)$ DISTRIBUTIVA

$(6 + 6) \cdot 5 + (-60)$ BOTTONE CALCOLO MOLTIPLICAZIONI

$(12) \cdot 5 + (-60)$ BOTTONE CALCOLO ADDIZIONI

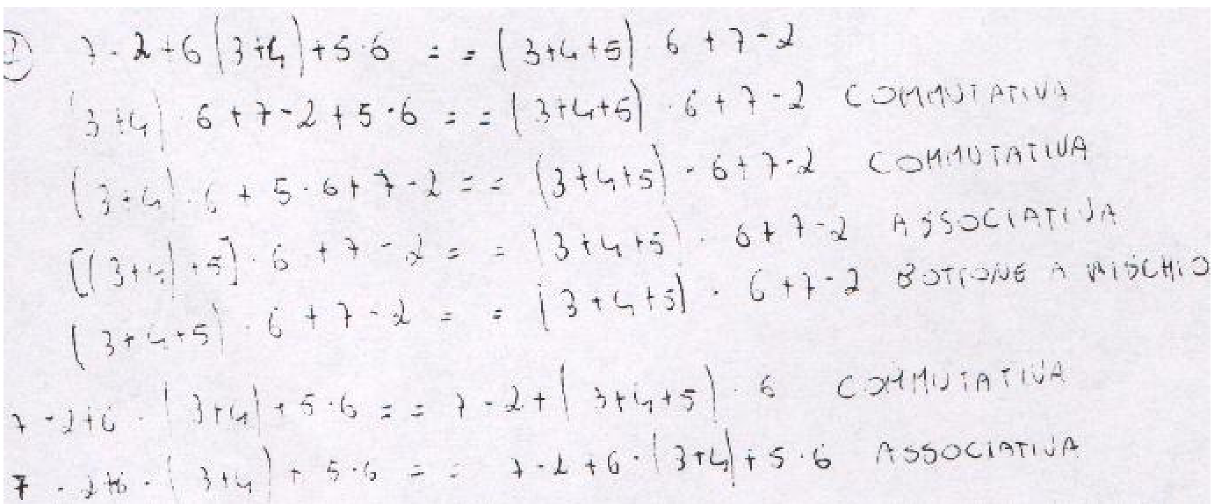
$60 + (-60)$ BOTTONE CALCOLO MOLTIPLICAZIONI

0 BOTTONE CALCOLO ADDIZIONI

Protocol 2: Nicola (class 1998, grade 9), for each step, highlights with a marker the sub expression where he thinks the button was applied.

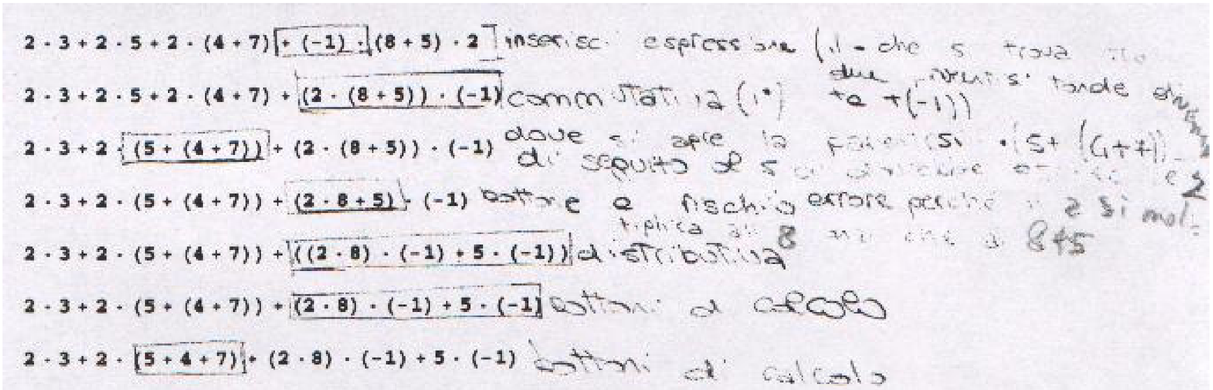
The pupil, in order to indicate what piece of expression each button was applied to, uses a marker to highlight the chosen part, producing a mark which is very similar to how the effect of a selection looks like in L'Algebrista. For instance, in L'Algebrista, the first selection performed in the case of Protocol 2, would appear exactly as "**3 • (2 • 5) + (2 • 3) • 5 + (-60)**", this suggests that the selection sign introduced by Nicola, comes directly from the selection tool of L'Algebrista, keeping some characteristic of its appearence.

this sign, clearly derived from the practice within L'Algebrista, is used by Nicola only in the first activity, which recalls directly what was done in the microworld. In fact, in the case of the second task, the pupil behaves differently: he did not use any sign interpretable as a "paper and pencil" counterpart of the selection tool, as shown in the Protocol 3. Nicola correctly transforms the leftomost expression into the rightmost, however, we observe that he skips some steps, in the sense that when he applies the commutative property, he applies at once, that of the sum and that of the multiplication (see steps from line 1 to line to and from line 5 to line 6). In other words, even though he recalls explicitly the practice of L'Algebrista (se for instance the use of the symbol "= =" and writing "Bottone a rischio"), Nicola syntesises transformational steps that probably are obvious for him, even if they are not for L'Algebrista.



Protocol 3 The protocol produced by Nicola (class 1998, grade 9) for activity CS2 doesn't present any attempt to highlight the sub expressions where the used transformation principles are applied.

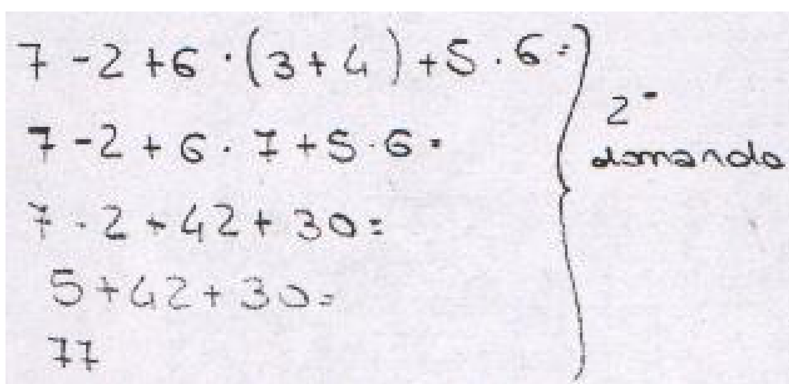
as far as the first activity is concerned, similar behaviour can be found in all the students' productions, a variety of signs are created representing the selection tool; for instance, Serena (Protocol 4), highlights "selected" expressions, imitating L'Algebrista, but differently from Nicola, she highlights the chosen sub expressions by inscribing them in a pencil drawn rectangle.



Protocol 4 Serena (class 1998, grade 9) indicates "selected" expressions by rounding them with hand drawn rectangles.

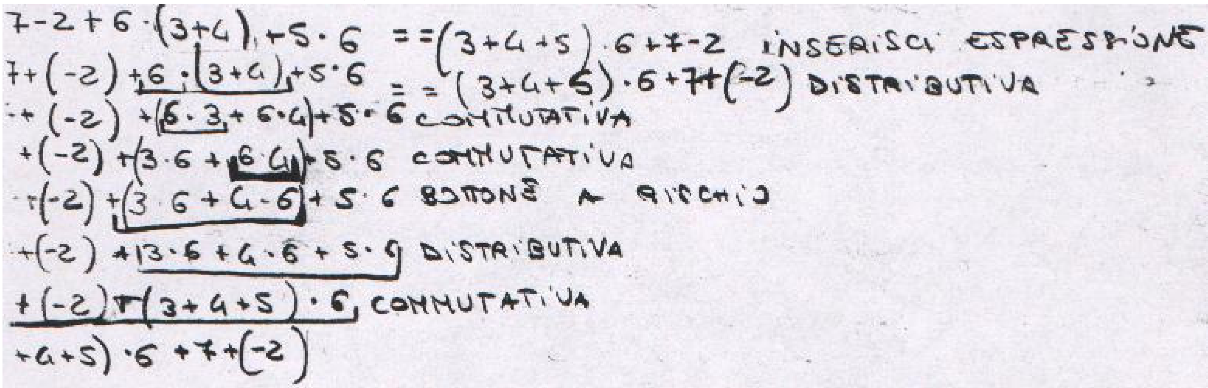
The second activity was not referred directly to the microworld, in fact in this case Nicola didn't do any reference to the selection tool, but he somehow referred to L'Algebrista by using the words "Bottone a rischio" (en: "Risky button") which is a clear reference to a button of the software (Protocol 3).

Most of Nicola's class mates did the same kind of reference to the commands of the microworlds, however, some of them, in the second activity, behaved significantly differently. It is the case, for instance, of Serena. she firstly checks the equivalence of the two expressions, as required, by simply computing each expression, following a standard procedure and a standard notation, as shown in the extract of Protocol 5. The pupil doesn't write any reference to the selection tool, coherently to the fact that computation of numerical expressions is a practice she was already well familiar with, before using L'Algebrista.



Protocol 5 Serena (class 1998, grade 9) computes the results of numerical expressions following a standard procedure and a standard notation.

Once checked the equivalencies by numerical computations, Serena proves them by means of axioms based transformations, as shown in Protocol 6.



Protocol 6 Serena (class 1998, grade 9), proves the equivalence of two expressions in the paper and pencil environment; for each step she writes the applied principle and highlights, underlying it, the sub expression where it is applied.

The pupil is for the first time involved in a proving activity within the paper and pencil environment, previously she has been experiencing proving equivalencies only once, in the computer laboratory, using L'Algebrista. The protocol she produced denounces the tranfer of a practice from the microworld into the paper and pencil environment: through the representation by derived signs a link is established between tools and practice in the microworl and signs and meanings in the paper and pencil environment, a link that will be elaborated in the following discussions under the guidance of the teacher. In fact, Serena uses several signs derived from L'Algebrista, such as the signs "= =", "BOTTONE A RISCHIO", and "INSERISCI ESPRESSIONE"[43]. These signs start to have a sort of double meaning, one referring to practices in L'Algebrista itself, and one referring to an extension of such practices to the paper and pencil environment, extension which has to be consistent with the algebra knowledge. We will discuss more in details the double nature of these signs in the next sections, here we concentrate on the particular sign produced by Serena and referred to the selection tool: for each line of the protocol we can see that a part of the reported expression is underlined; in other words, every time she applied a transformation rule to a sub expression of an expression, she "selected" the considered sub expression and highlighted it underlying it. This practice of underlying sub expressions, and its related signs, is clearly derived from the practice of transforming expressions in L'Algebrista, and from the selecting tool and has no counterpart in any sign related to previous computation practice: in fact, it is not present in the case of the simple numerical computation of this task (see for instance Protocol 5), However, even if the practice of selecting sub expressions is derived from L'Algebrista, it is interesting to observe that Serena in this case uses a sign which is different (one could say an evolution) from that used in the first activity to directly represent the selection tool. Whilest in the previous cases she drew whole rectangles, which looked more like the appearance of the selections in the microworld, here she underlines the sub-expression, carefully marking the beginning and the end .

This evloution of the sing, from directly representing the selection tool by a copy to evoking it through a symbol which keeps its characterising elements, namely marks for representing the beginning and the end of the selected expression. It is interesting to observe that the sign, clearly derived form the selection tool of L'Algebrista, doesn't have the same functionality, i.e. that of being necessary for activating commands. In fact, this new sign seems to have a double functionality: on the one hand it gives a support for recognising a sub expression to be substituted with an equivalent one, and on the other hand it communicate to the reader what is the sub

[43] en.: "Insert Expression".

expression to be transformed. Our interest in such genesis is due to the fact that it witnesses the emerging of signs, derived from the practice transformation in the micorworld, in a corresponding practice of transformationin paper and pencil. thanks to L'Algebrista a new practice introduced in the context of numerical expressions, doesn't seem to have cancelled, or transformed, the previously acquired computational practices. Yet, it clearly represents a starting point for a new practice in the paper and pencil environment, that we can recognize as consistent with the algebraic way to handle expressions.

Even if both the activities concerned proofs of equivalencies of expressions, the first one referred directly to the microworld, while the second did not, and this was reflected by the behaviour of Serena. In other words, we are assisting to the genesis of the practice of proving equivalencies in paper and pencil, a practice that is new for Serena and that is derived from the activity of proving equivalencies in L'Algebrista. The pupil inherits behaviours and signs from the practice in the microworld, and employs them to bring forward the new practice in paper and pencil, but the two practices are not totally identified, as witnessed by the use of different signs for selecting sub expressions. In fact, the practice of highlighting sub expressions, itself, has different meanings in the two environments. In L'Algebrista it is an unavoidable step, which is necessary if one wants to apply commands on the expression he/she is working with, and at the same time it can be interpreted as a mean for communicating with the computer at the interface level. In paper and pencil the practice of highlighting sub expressions is avoidable if the goal is only to transform expressions (as Nicola did in Protocol 3), but it can be both an effective means for communicating details on the transformation process that is going on, and an effective means for keeping the control on the transformation performed. From this point of view it is interesting the case of Serena, who avoided using this "selection sign" in the case of standard computations, but employed it in the context of proving, where it was required to communicate relevant information such as the sub expressions to which each transformation principle was applied. This suggests the presence of a double interpretation of expressions, on the one hand they are interpreted a computation procedure to be executed (in the case of equivalence checking, see Protocol 5), on the other hand it is possible to look at the structure of the expression and extract from it a sub expression highlighting it (in the cases of equivalence proving either in the computer microworld, either in paper and pencil, see Protocol 4 and Protocol 6). In other words, the written productions of the pupils show evidence of the fact that expressions begin to be interpreted not only as processes, but also as objects, made of other smaller objects that can be identified; we believe this to be an important step toward a structural interpretation of expressions. Such results, as showed by the examples, seem to be relatedto the selection tool, which at this point can be used by the teacher as an instrument of semiotic mediatation during the collective disussions.

Behaviours similar to that of Serena, can be found in many of the protocols produced by the other pupils, giving birth to a new attitude that later on, after being shared by means of mathematical discussions, became a use of the whole class. The attitude is that of underlying (or "selecting" by means of other graphic tools) sub expressions, whenever given transformation rules have to be applied to them. To sum up, thanks to the selection tool, some pupils originated new signs associated to the activity of comparison, these new signs were then shared with, resulting in an set of signs (each pupil used his/her signifying forms) all referring to the common meaning of sub-expression.

*9.2.3.2.**The autonomy of the selection sign***

The use of the selection sign[44], however, turns out to be not only a mean for communicating, but a real tool available to pupils as an instrument for managing expressions, providing them an

[44] From now on this is how we will refer to the practice of highlighting sub expressions of a given expression by means of some graphic tool.

external control of what they are doing. Interesting examples can be found in situations where pupils are coping with objects and activities that are new for them. It is the case, for instance, of the first activity with letters, in which pupils are required to conjecture which of a set of given literal expressions can be interpreted as a correct formula for computing the area of a given geometrical figure (see activity CL 7 in appendix 13.3.). the activity was carried out in the computer laboratory, pupils are left free to work either using L'Algebrista or with paper and pencil, as they prefer. Moreover by that moment, in the microworld, they are working with "theory one", which includes very few buttons: computation buttons, buttons of the properties of the operations, and buttons of the properties of neutral elements, as shown in Figure 16.

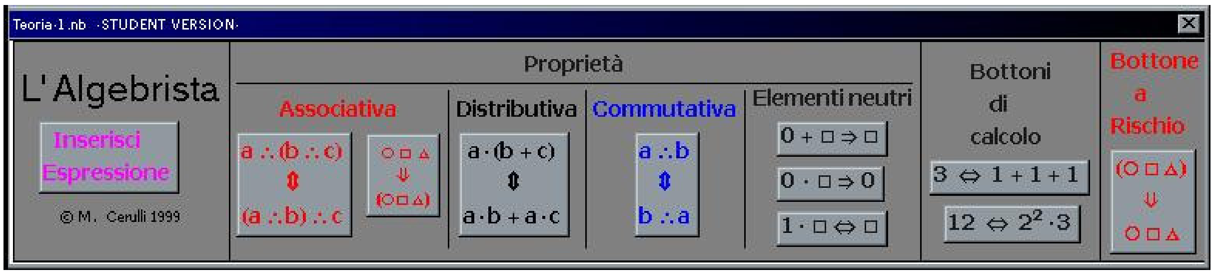
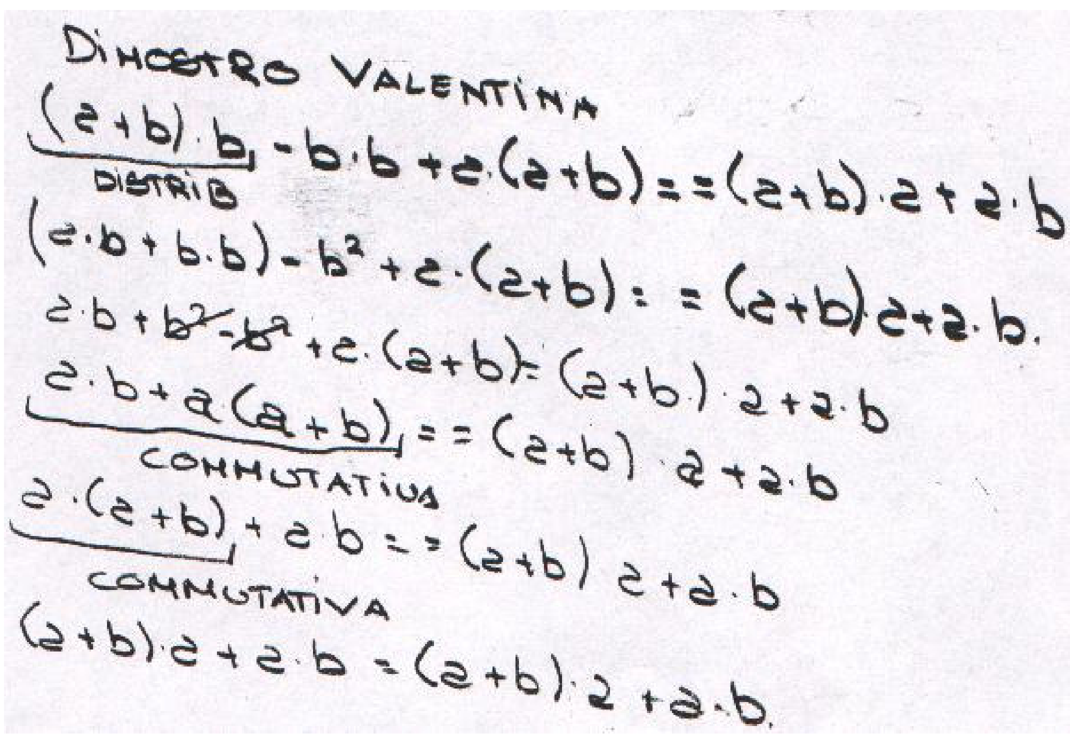


Figure 16 The palette of "Teoria 1", i.e. "Theory one".

We consider the case of Eleonora (class 1999) who first produced her conjecture by reasoning geometrically, then, in order to justify her conjecture (they were not required to prove it), she begins by "proving"[45] by means of geometrical arguments that one of the given expressions is correct (see Protocol 8), and then she "proves" the correctness of the other expressions by transforming them into the first one (see Protocol 8).

1
2
3
4
5
6



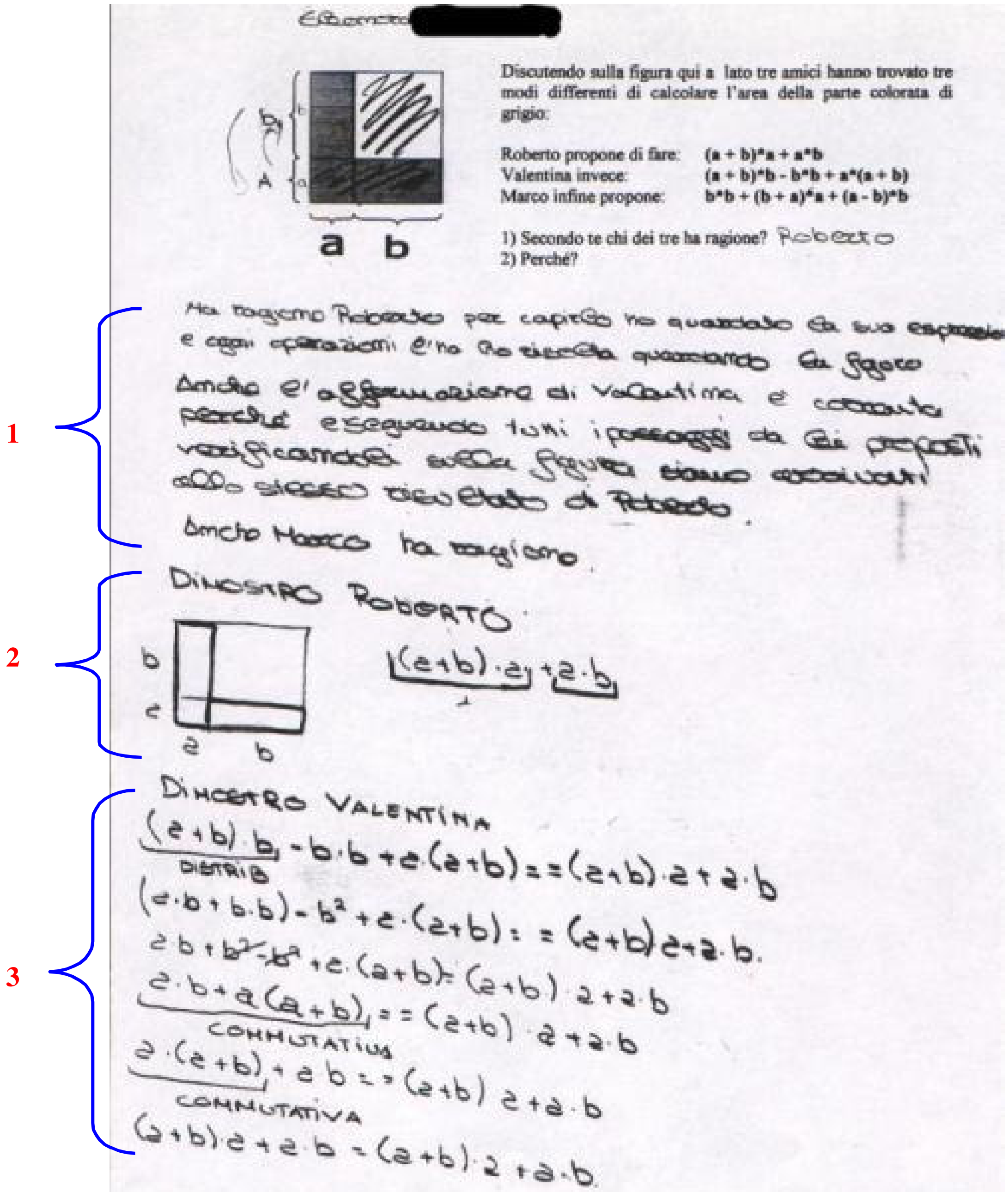
Protocol 7 Eleonora (class 1998, grade 9) transforms the expression on the left into the expression on the right. Whenever she uses a formalised transformation rule, she uses the selection tool to individuate were to apply it.

[45] Eleonora actually usese the word "dimostro" which means "I prove".

As already said, this is the very first experience the pupils have with literal expressions, and Eleonora finds herself in the need of executing two kinds of transformations: on the one hand transformations corresponding to available buttons of L'Algebrista and formalised in terms of the properties of the operations (the axioms of a theory); on the other hand transformations that are not yet formalised in class practices (and that do not correspond to any available button of the microworld). In order to accomplish her task, Eleonora finds herself in the need of using also the second kind of transformation, with which she is somehow familiar, in spite of the fact that they have not yet been shared and formalised by the class in the form of accepted axioms and corresponding buttons of the microworld. The transformations that transform line 2 into line 3 and line 3 into line 4 belong to the second category, while the others to the first. The distinction between the two kinds of transformations is very clearly expressed in terms of the used graphic signs, in fact in the first case, and only in that case, Eleonora uses the selection sign and reports the property used. In the other case, in the transformation form line 2 to line 3 she doesn't write anything, while from line 3 to line 4 she uses standard signs for deleting terms of expressions by barring them.

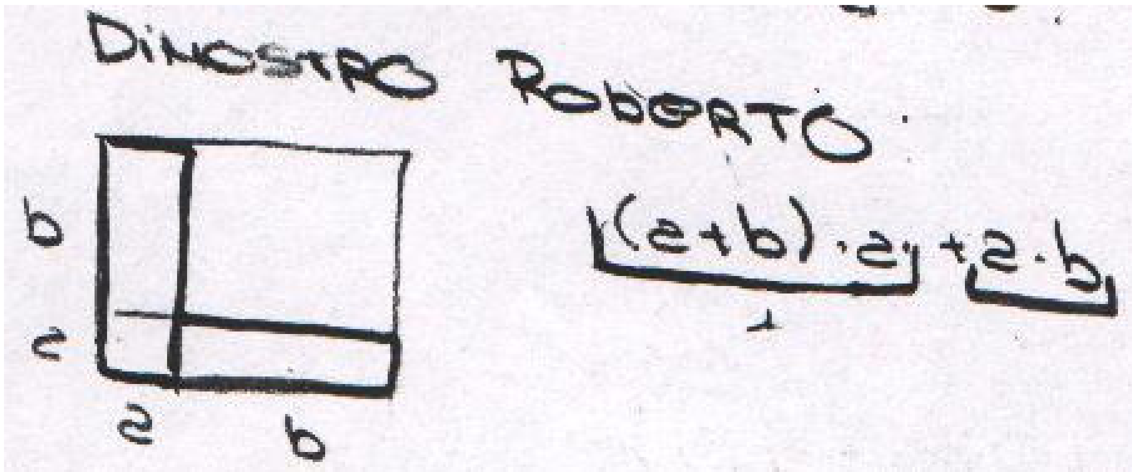
Eleonora is familiar with the selection sign, and with the properties of the operations, as means for proving equivalencies with numerical expressions. The protocol shows that she is able to extend the practice of equivalence proving to the case of letters, in fact she successfully selects expressions and correctly applies properties of the operations. The pupil works on the structure of the expression, by selecting sub parts, and transforming them according to the axioms, showing that the new objects, literal expressions, are handled in the same way as old objects, numerical expressions, according to the similarity of their structures.

Finally, the fact that, in the case of transformations that do not correspond to available buttons or to axioms accepted by the class, Eleonora doesn't use the selection sign, may suggest that this sign is not completely independent and is interpreted only as one of the elements constituting the practice of transforming expressions by means of axioms or buttons. However, the behaviour of Eleonora in her geometrical "proof" (Protocol 9) suggests that in certain situations the pupil is able to interpret the selection sign as an istrument to be used to accomplish a task, independently from the practice of transforming expressions, and without references to buttons or axioms; in other words, it is a sign which can be used as a mean for expressing the structure of an expression.



Protocol 8 Eleonora (class 1998, grade 9) first produced her conjecture by reasoning geometrically (1), then, in order to justify her conjecture, she provides geometrical arguments that one of the given expressions is correct (2), and finally she "proves" the correctness of the other expressions by transforming them into the first one (3).

In fact, in the case the geometrical proof, Eleonora splits the considered expression in two parts, and then interprets these two sub expressions as the formulas for computing the areas of two small rectangles composing the given figure. What is very interesting is that in order to split the expression, Eleonora simply "selects" the two considered sub expressions using the same graphic sign that she uses for selecting sub expressions in transformational activities (see Protocol 9).



Protocol 9 Eleonora (class 1998, grade 9) uses a sign derived from the selection tool of L'Algebrista, as a mean for splitting an expression into two sub expressions to be interpreted as formulas for computing the areas of two sub rectangles of the picture on the left.

In other words, the sign derived from the selection tool, and transferred in the paper and pencil environment to represent a sub-expression to be transformed, has lost its function in relation to the transformation, keeping the meaning of representing the sub-structure of the expression. The sign is used by Eleonora both as an instrument for acting upon the structure of the given expression, splitting it into two parts, and as an instrument for communicating how the expression can be split in two parts. The example shows that the selection sign reached autonomy from the context in which it was originated (that of transforming expressions), and is used coherently also in a different context, out of the sphere of practice that originated it. The new use of the sign is consistent with the old one, in fact the sign here highlights the structure of the given expression identifying two sub expressions, however its function is not that to allow the application of a button, but it is to represent a decomposition of the expression consistent with the decomposition of the figure.

9.2.4. Symbols and meanings of equivalence

The issue of equivalence of expressions is central to our approach to symbolic manipulation, which is based on the idea of transforming expressions by means of axioms that keep equivalencies. Due to this centrality, each of our experimentation begins with a collective discussion aiming at reaching an agreement on what it means that two expressions are equivalent. Such meaning, however, is expected to develop, thus often during the experimentation the class needs to re-discuss and re-shape it in order to take into account new practices experienced by pupils. Here we are going to trace some steps of the evolution of this meaning highlighting the role played by L'Algebrista in mediating and shaping it.

*9.2.4.1.**Preliminary definitions of equivalence***

The first class discussions of the experimentation are focused on the meaning of the word "equivalent", aiming at sharing of a definition of the equivalence of expressions. Such discussions highlights several meanings related to the words "equivalente" (en.: "equivalent") and "uguale" (en.: “equal”), meanings that are exploited by the teacher to reach an agreement on a shared meaning coherent to her intentional knowledge. The teacher exploits the multiplicity of meanings to guide the class to reach an agreement on the definition of equivalence of expressions (in terms of equality of results). However, pupils could not reach an agreement on the definition of equal expressions.

Along the discussion the teacher exploited the polisemy of symbols such as the written "=" and the spoken words "equal" and "equivalent". After the discussion, as a homework, pupils are required to write a report of the discussion.

Protocol 10 is an exemplary case of text produced by a pupil, giving an idea of what kind of discussion had been going on in class.

1. *When is it that two expressions are equal?*
2. *We discussed about it together in our classroom and it turned out that two expressions with the same numbers and the same operations are equal.*
3. *Es.*
4. *5+3=5+3*
5. *We said that two expressions that lead to the same result are equivalent*
6. *5+3=5+3*
7. *But then we doubted if 5+3=3+5 is equal or equivalent; concerning equivalent we are all sure; but 5+3=3+5?*
8. *Someone said that it was equal, whilst others said that it was only equivalent: so a question raised:*
9. *If we apply a property of the operations to an expression, do we obtain equal or equivalent expressions?*
10. *The sure answer was equivalent, because 5+3 gives 8 and 3+5 is still equal to 8.*
11. *On the basis of what we had said, that is, two expressions are equal only if they are identical, then the writing*
12. *5+3=4+4 is wrong because the symbol equal is used in the wrong way*
13. *And we arrived to state that the symbol = doesn't always distinguish between equal and equivalent;*
14. *but we didn't succeed reaching a good agreement on when two expressions are equal.*

Protocol 10 Valeria (class 1999, grade 9), writes a report on a mathematical discussion concerning the words equal and equivalent. The lines have been numbered by the author of this dissertation.

Valeria's report illustrates how passionate was the discussion that went on in the class and highlights a confusion concerning the meanings of the words "equal" and "equivalent" which are interpreted as distinct, although often represented by the same symbol "=". The symbols/words “=”, “equal” and “equivalent”, share some meanings, as witnessed by Valeria (line 13) who highlights the fact that the symbol "=" can stand for both "equal" and "equivalent". We presented Valeria's protocol as an example of the initial fertility of the territory of the idea of equivalence, a territory in which the teacher planted the seed of a theroretical perspective for symbolic manipulation: the idea of comparing expressions and discussing the criteria for establishing an equivalence realtionship, among them, to have the same result and that of applying the properties of the operations (see lines 9 and 10).

Other pupils highlighted other salient episodes of the discussion, corresponding to other key elements of the ideas of equivalence of expressions and transformations by means of the properties of the operations. Let's consider the following excerpt of the report produced by Veronica, Protocol 11.

1. *[…] we wrote an expression and applied several properties of the operations, thus solving it into its new forms and we saw that each is equivalent to the others, but we didn't understand*

if they are also equal.

2. *After having written an expression like this: 2,(3+5)+2,2 and having applied the dissociative property of the multiplication with respect to the sum (2,3+2,5)+2,2, there was a new observation: not only the first expression is equal to the second one, but also the second one is equal to the first one*

Protocol 11 An excerpt of Veronica's (class 1999, grade 9) report on the discussion concerning the meanings of the words equal and equivalent. The lines have been numbered by the author of this dissertation.

Veronica (line 1) highlights the fact that the properties of the expressions can be used to obtain new "forms" of a given expression, which are all equivalent; moreover (line 2), she reports how they indirectly faced the reflexivity of equivalencies and the reversibility of transformations.

The symbols and the word she uses show the existent confusion, for instance in line 2, she uses the word "equal" probably meaning "equivalent". This kind of confusion is also present in the protocols of most of her class mates. However, two things seems to be clear, the idea of equivalence in terms of computed results (Valeria, Protocol 10, line 5), and the fact that equality implies equivalence as witnessed by Tiziano's assertion (Protocol 12):

"[…] Others claim that two expressions are equal if they are equivalent (have the same result). The only certainty that we had is that if two expressions are equal, then they are also equivalent".

Protocol 12 An excerpt of Tiziano's (class 1999, grade 9) report on the discussion concerning the meanings of the words equal and equivalent.

To sum up, pupil's reports show that the discussion highlighted overlapings and differences concerning the meanings of the symbols/words "=", "equal" and "equivalent", such symbols result to be polysemic, in the sense that often one is used as standing for the others. At this point the the reference to one or another of their meanings, is rather instable and mainly unconscious; one of the objectives of the teaching intervention will be that of constructing a stable and conscius polysemy related both to the symbols "=" an the symbols "equal".

*9.2.4.2.**The role of the symbol "= =" derived from L'Algebrista***

On the basis of the outcomes of discussions like the one reported in pupil's protocols in the previous paragraph, the meanings related to the concept of equivalence are shaped along the experimentation by means of practices involving also L'Algebrista.

When the microworld is introduced in class practices as a means for comparing expressions by checking and proving their equivalence relationships, a new symbol is introduced, that of the double equal sign "= =" (a double "=") which can be used as a separator between two expressions, allowing the user to insert them simultaneously in the microworld, as shown in Figure 17. The idea of using the symbol "= =" is suggested to pupils by the teacher, on purpose, and justyfied also by the fact that the symbol "=" cannot be used in L'Algebrista, because it has another meaning. Actually the symbol "= =" was originated by technical needs, but resulted to be good choice exactly because it originated a new symbol to be used by pupils.

A key step is that of introducing, on purpose, the symbol "= =" as a means for breaking the polysemy of the symbol "=" (discussed in the previous paragraph), and to distinguish meanings related to "equality" from the meanings related to "equivalence". The new symbol "= =" is introduced as standing for "equivalent", so that the idea of equivalence can be built in relation and as opposed to the idea of equality represented by the symbol "=". However, this distinction is intended to be temporary. At the end, both the ideas of equivalence and equality are going to be condensed in the sign "=", when its polysemy can be managed by pupils because its meanings had been well stated in the practice of equivalence proving.

7 – 2 + 6 * (3 + 4) + 5 * 6 == (3 + 4 + 5) * 6 + 7 – 2

■ 7 + (–2) + 6 · (3 + 4) + 5 · 6 == (3 + 4 + 5) · 6 + 7 + (–2)

Inizio 7 + (–2) + 6 · (3 + 4) + 5 · 6 == (3 + 4 + 5) · 6 + 7 + (–2)

Figure 17 In L'Algebrista it is possible to insert two expressions one next to the other, separated by the symbol "==". Here the first line is the text written by the user, and the other lines are produced by the software after clicking on the button "Inserisci Espressione" (en.: "Insert Expression"). In particular, the third line, the one with the blue label "Inizio" (en.: "Start"), is situated in the space of the microworld where it is possible to transform expressions using buttons, it is upon this line that users commands are to be applied.

Once two expressions are inserted as separated by the symbol "= =", then the user can indifferently transform the leftmost expression or the rightmost expressions, obtaining a new line still made of the two expressions, with the resulting modifications, as shown in Figure 18 and Figure 19.

Inizio 7 + (–2) + 6 · (3 + 4) + 5 · 6 == (3 + 4 + 5) · 6 + 7 + (–2)

a.·.b⇔b.·.a → 5 · 6 7 + (–2) + 6 · (3 + 4) + 6 · 5 == (3 + 4 + 5) · 6 + 7 + (–2)

Figure 18 It is possible to transform the left most expression by selecting it and clicking on a button, the new obtained line is made of the modified left expression and the unmodified right expression.

Inizio 7 + (–2) + 6 · (3 + 4) + 5 · 6 == (3 + 4 + 5) · 6 + 7 + (–2)

a.·.b⇔b.·.a → 5 · 6 7 + (–2) + 6 · (3 + 4) + 6 · 5 == (3 + 4 + 5) · 6 + 7 + (–2)

a.·.b⇔b.·.a → (3 + (4 + 5)) · 6 7 + (–2) + 6 · (3 + 4) + 6 · 5 == 6 · (3 + (4 + 5)) + 7 + (–2)

Figure 19 It is possible to transform the right most expression by selecting it and clicking on a button, the new obtained line is made of the modified right expression and the unmodified left expression[46].

In this way, if the aim is to transform one expression into the other, or both into a third equivalent one, it is always possible, step by step, to modify one of the expressions using the other expression as a target to point at. It is then possible to obtain a chain of lines ending up with a line where the two expressions look exactly the same, as shown in the following example.

[46] In the example, the brackets are added automatically by L'Algebrista, this is a bug of the software with which pupils learn to cope easily by using the "Risky Button" (ita.: "Bottone a Rischio") to get rid of unwantd brackets.

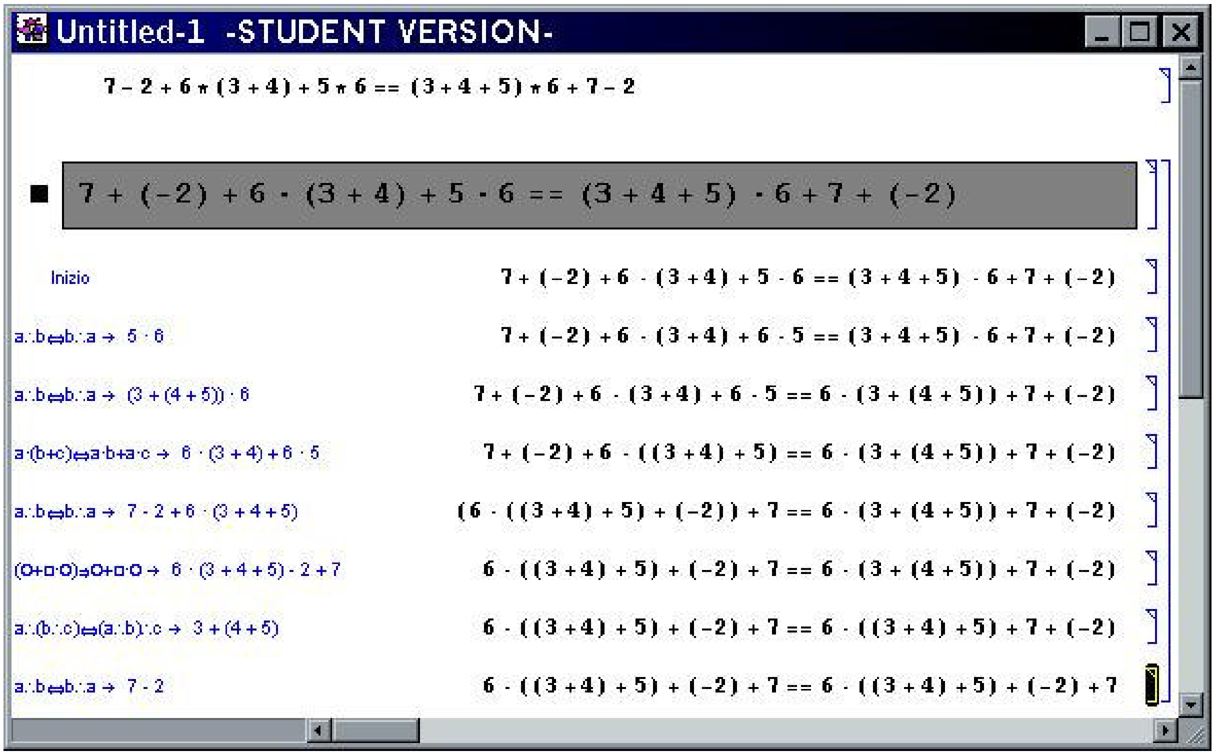


Figure 20 At each step it is possible to transform one of the two expressions pointing using the other expression as a target to aim at. In the case of equivalence proving, the chain of transformations end when a line with two identical expressions is obtained.

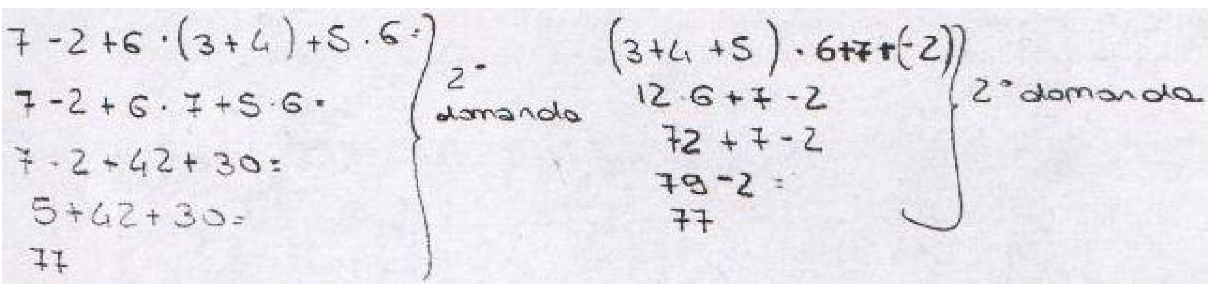
The sign "= =" is introduced on purpose by the teacher when pupils are proposed the first activities of equivalence proving with L'Algebrista, shaping such practices toward a structural perspective. The example shown in Figure 20 is paradigmatic: in L'Algebrista, thanks to the sign "= =", the activity of proving equivalencies can be interpreted in a very particular way. First of all to write two expressions separated by the symbol "= =" becomes a way to declare the intent of proving the supposed equivalence of the expressions. Secondly, starting from the line "**A == B**" (where A and B are expressions), and transforming **A** by means of the properties of the operations, a new line "$\mathbf{A_1}$ **== B**" is obtained where $\mathbf{A_1}$ is equivalent to **A** because of the used axiom. The transformation process ends up when we obtain "$\mathbf{A_n}$ **==** $\mathbf{B_m}$" where $\mathbf{A_n}$ is identical to $\mathbf{B_m}$. The process has a clear start, marked by the actof inserting "**A == B**" in the microworld by clicking the "Insert Button" and has a clear end, when an identy is reached. At the beginning of the process the symbol "= =" is a declaration of intent of proving the supposed equivalence, and at the end of the process it can be interpreted as a statement of such equivalence, on the basis of the produced proof. During the process the equivalence relationship of the two expressions is pending, and can be declared proved only after the end of the process. Nevertheless, the symbol "= =" is used also in the intermediary phase. To sum up a writing such as "**A = = B**" has at least three different meanings: "declaration of intent of proving the equivalence of **A** and **B**"; "pending equivalence of **A** and **B**" (during the proving process); "statement of equivalence of **A** and **B**".

The meanings of the symbol "= =" become strictly related to both the equivalence by means of axioms, and the equality as identity; in fact, at the end of the proving process the two final expressions are identical, thus equal, thus the initial expressions are equivalent. The symbol introduced on purpose in class practice, concerning equivalence proving, brings with itself meanings originated in the practice with L'Algebrista, and enriches the fertile territory that we described in the previous section concerning meanings and symbols related to the idea of equivalence (see 9.2.4.1)..

The relationship between the sign of equality ("=") and the sign of equivalence ("= ="), is established in relation with the practices triggered by the use of the sign "= =". In fact, such sign firstly stands for "pending equivalence" functioning as a stimulus for action, and at the end, when the two expressions become equal, we have a stop condition constituted by the obtained identity. The opposition between "go further on" and "stop" is determined by the opposition between "equal" and "equivalent", pending equivalence corresponds to "go further", whilst equality stands for "stop", and after the stop condition the equivalence is not pending any more.

We are now going to illustrate these considerations drawing from pupils protocols. We start going back to the activity **CS 2** that we considered in section **9.2.3**, a homework (thus the computer was not available) which required pupils to compare two numerical expressions, checking and proving their equivalence. In particular pupils were asked, to produce two proofs, the first transforming the first expression into the second one, and the latter transforming the second expression into the first one.

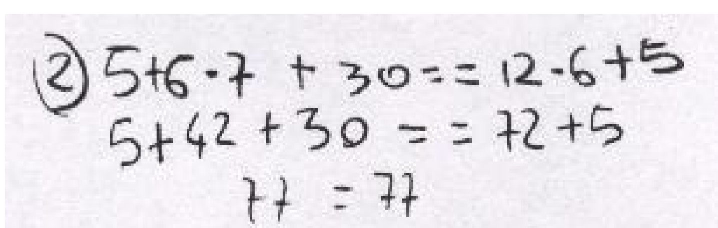
In the phase of checking the equivalence by means of numerical computations, most of the students simply produced a standard computation, without using any symbol derived from L'Algebrista, like for instance the case of Serena (see Protocol 13). Serena checks the equivalence of the given expressions by computing them separately, using a standard procedure and a standard notation. The two computational procedures are both represented as a chain of steps, most of the step are separated from the following ones by the symbol "=", meaning that each step is obtained from the precedent one. Thus the symbol "=" here is used with a strong directionality, because the a computation procedure follows a flow going from the given expression to its numerical result, passing several times trhough the symbol "=".



Protocol 13 Serena (class 1998, grade 9) checks the equivalence of the given expressions by computing them separately, using the standard procedure and notation for computing expressions. The two computational procedures are both represented as a chain of steps separated by the symbol "=", which, in this case, means that each step is obtained from the precedent one. Thus the symbol "=" here is used with a strong directionality.

Differently from Serena, other pupils introduced the new symbol "= =", it is for instance the case of Marta, Protocol 14.

1
2
3

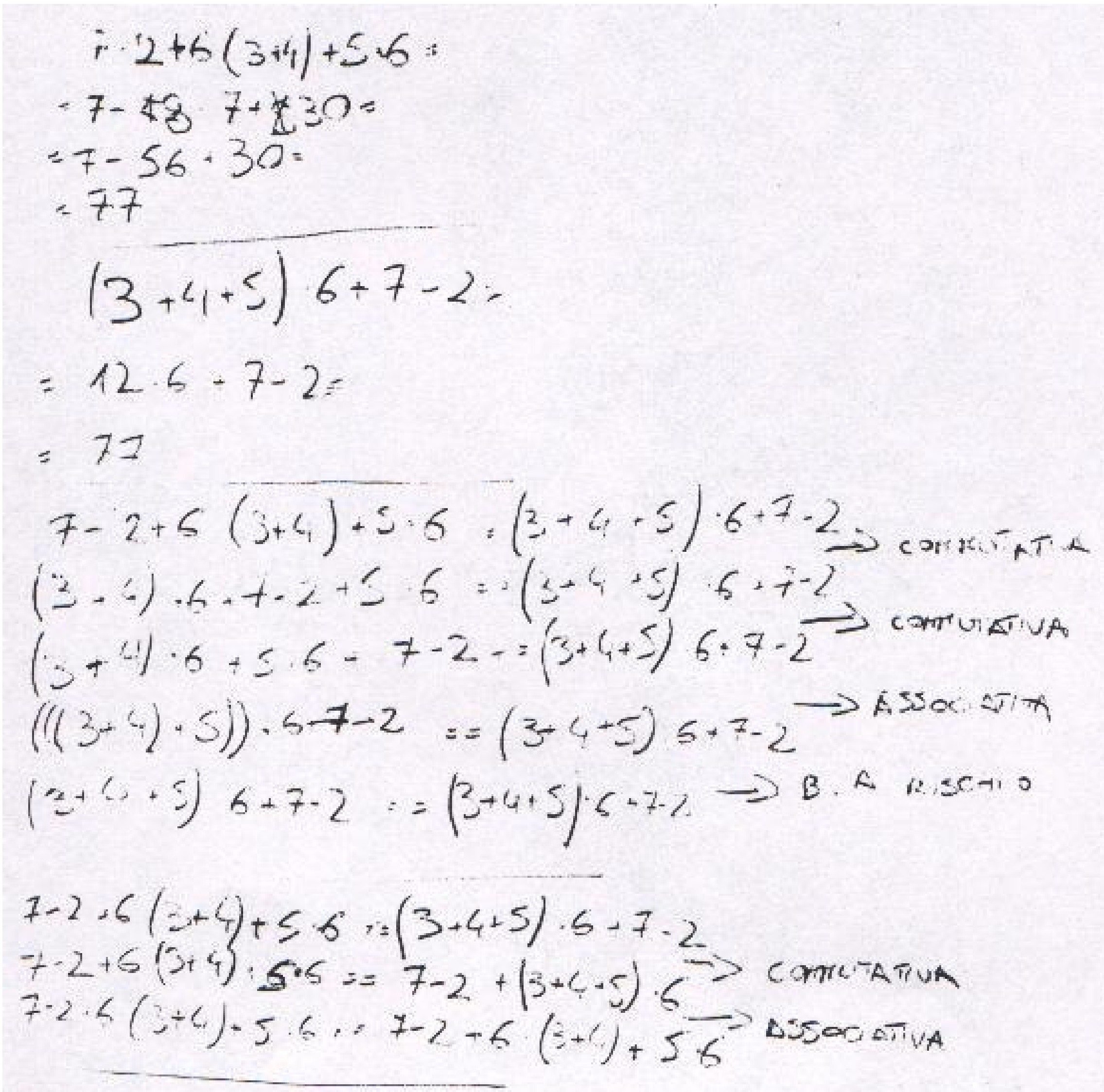


Protocol 14 Marta (class 1998, grade 9), in order to check the equivalence of the expressions "**7-2+6¸(3+4)+5¸6**" and "**(3+4+5)¸6+7-2**", computes them in parallel, separating the two computation procedures by means of the symbol "= =" (lines 1 and 2). Only at the end (line 3), when the two obtained expressions are actually identical, she substitutes the symbol "= =" with the symbol "=".

The task requires pupils to check the equivalence of the expressions **"7-2+6·(3+4)+5·6"** and **"(3+4+5)·6+7-2"** by computing them. Marta executes the computational steps of the two expressions in parallel (lines 1 and 2), writing them one next to the other, separated by the symbol "= =": the left side represents the computational steps of the first expressions, while the right side represents the computational steps of the second expression. The protocol produced by Marta clearly recalls the practice of comparing expressions in L'Algebrista, and in the first two lines the symbol "= =" is probably a symbol expressing a pending equivalence. For sure "= =" is not used as a mean to state equality, in fact, only in the last line (3), the one with the numerical results, Marta uses the symbol "=", instead of using the sign "= =". In this protocol the symbol "=", differently from the case of Serena's protocol, doesn't have a preferencial direction, and it is used to state an equality relationship between the two results, at the end of the process of computation, coinciding with the end of the comparison.

The practice of comparing expressions in L'Algebrista, is extended by Marta to the paper and pencil environment, as witnessed by her use of the symbol "= =", and by the lay out of her protocol. As a consequence, the symbol "=", seems to have inherited from the symbol "= =" the property of being non directional (as opposed to the strong directionality found in the protocol of Serena Protocol 13); in this sense, the symbol "= =" enriches the polysemy of the symbol "=" with meanings derived from practices situated in L'Algebrista. In particular in this cases the symbol "=" seems to have inherited, from the symbol "= =", the property of being, at least in this situation, non directional. This property is typical of a structural interpretation, as opposed to its strong directionality in operational perspectives.

At the beginning of the experimentation pupils were already familiar with computing numerical expressions, this may explain why only in isolated cases, like that of Marta, they used the symbol "= =" in their computations. However, the activity **CS 2** required pupils also to prove the equivalence of the two expressions, by means of axioms (or buttons) based transformations. Such a practice is very new for pupils, and actually they had experienced it only in the L'Algebrista. In fact, in this case all the pupils referred to the microworld, by using signs directly derived from it. In particular, some of them, introduced in their paper and pencil practice the symbol "= =", as can be seen in the protocols like that of Nicola (**Protocol 3**) and that of Serena (**Protocol 6**), or that of Roberto in (Protocol 15).



1 7·2+6(3+4)+5·6 =
2 = 7-48·7+30=
3 = 7-56+30=
4 = 77
5 (3+4+5)·6+7-2=
6 = 12·6+7-2=
7 = 77
8 7-2+6(3+4)+5·6 = (3+4+5)·6+7-2 → COMMUTATIVA
9 (3+4)·6+7-2+5·6 = = (3+4+5)·6+7-2 → COMMUTATIVA
10 (3+4)·6+5·6+7-2 = = (3+4+5)6+7-2
11 (((3+4)+5))·6+7-2 = = (3+4+5)6+7-2 → ASSOCIATIVA
12 (3+4+5)6+7-2 = = (3+4+5)·6+7-2 → B.A RISCHIO
13 7-2+6(3+4)+5·6 = = (3+4+5)·6+7-2
14 7-2+6(3+4)+5·6 = = 7-2+(3+4+5)·6 → COMMUTATIVA
15 7-2+6(3+4)+5·6 = = 7-2+6·(3+4)+5·6 → ASSOCIATIVA

Protocol 15 Roberto (class 1998, grade 9) first computes the results of the given expressions by computing them (lines l-4 and 5-7). He uses the equal sign to mark the passage from one computation step to the other (lines l-4 and 5-7), thus the sign results to be oriented from left to right. On the other hand the proofs (lines 8-12 and 13-15) of equivalence proposed by Roberto has a layout similar to that of the proofs produced in L'Algebrista, and he uses the symbol "= =" as a non directional sign (lines 8-12 and 13-15), as opposed to the directionality of the symbol "=" in the first part of his protocol (lines l-4 and 5-7).

As a matter of fact, the word "equivalent" results to be associated to both symbols, "= =" and "=", resulting into a polysemy derived from different contexts of practices. Such polysemy can be exploited by the teacher in order to build specific aspects of the meaning of the word "equivalent" that come from different spheres of practice. In particular, as showed by the example of Marta (Protocol 14), one apsect of the meaning, such as that of the bidirectionality of equivalencies, can be derived from the practical context of L'Algebrista. The microworld provides the potentialities, and teacher's guidance together with the purposeful introduction of the symbol "= =", aim at leading pupils to reach control on the meanings related to the symbol "=" and to the ideas of equality and equivalence.

9.2.5. Pupils controlling symbols and meanings of equivalence

In the previous sections we firstly presented protocols taken from the beginning phase of respective teaching experiments, then we presented protocols taken from a key intermediate phase, that of the introduciton of literal expressions. At that stage pupils are not yet completely able to manage all the different meanings and symbols related to the idea of equivalence, however, the examples that we presented show how symbols and meanings are being shaped towards a structural perspective. Going on with the sequence of activities, pupils' mastery of the idea of equivalence, and related symbols, increases. Evidence is provided by the answers given by pupils of class 2003 to a dedicated test (see appendix **13.8.**) that we subtmitted to them at the end of the experimenation, after they had just approached the problem of solving equations. In Protocol 16 and Protocol 17 we report some excerpts of the answers produced by Elisabetta and Daniela.

Question	Elisabetta's answers
1. The symbol "=" a. What do you think the symbol "=" means in algebra? b. Write examples using this symbol	*"From the beginning of this school year, till now, with our teach and my classmates, we established that the symbol '=' means 'EQUAL'. For instance if are given the expressions '5+3-2,7' and '5+3-2,7' we can write '5+3-2,7=5+3-2,7' because they are equal. For us, the word 'EQUAL' means 'identical', thus, even if, for instance, two expressions give the same result, but initially are not 'identical', we cannot state that they are equal. For instance: '2,3+2,5' and '2(3+5)' give the same result (16), but are not identical."*
2. The word "equal" a. What do you think the word "equal" means in algebra? b. Write examples using this word	
3. The word "equivalent" a. What do you think the word "equivalent" means in algebra? b. Write examples using this word	*"Still basing on what I have learnt in these months, the word 'equivalent' means 'having the same result'. For instance if I consider the 2 expressions: '2,3+2,5' and '2(3+5)', even if they are not identical, I can notice that they have the same final result, or I can notice that if I apply the AXIOM OF THE DISTRIBUTIVE PROPERTY I can render them 'EQUAL'; thus they are 'equivalent' because they have the same final result or because by applying an axiom I can 'transform them' untill they are 'identical'. Sure enough, in the case of literal expressions which are not initially equal, I can prove their equivalence only by trying to render them equal, and thus equivalent."*
4. Other symbols and words a. What relationship do you think there is between the symbol "=", the word "equal" and the word "equivalent"? You can eventually show examples	*"In our class, the symbol '=' is used to indicate that two expressions are "equal", while the symbol "= =" is used to indicate that two expressions are equivalent. Of course when I succed, by means of the*

b. Write other symbols and words that you know, and you think are related to the symbol "=" or to the words "equal" and "equivalent". You can eventually show examples	*application of theorems and axioms, to transform an expression into an equivalent one, at the end I obtain two expressions that apart from being equivalent, they are also equal, and in this case both the symbols "=" and "==" can be used."*

Protocol 16 Elisabetta's (class 2003, grade 9) answers to the first four questions of the final test (see appendix **13.8.**).

Question	Daniela's answers
1. The symbol "=" a. What do you think the symbol "=" means in algebra? b. Write examples using this symbol	*"a) I think in algebra the symbol '=' has the meaning of equality, that is, it indicates that two elements that it compares are equal, identical. […]"*
2. The word "equal" a. What do you think the word "equal" means in algebra? b. Write examples using this word	*"a) I think that in algebra "equal" means identical. I think that an element is equal to another if they are identical.* *b) This word is used in daily life. For instance 'your pen is equal to mine' and the two pens are identical, even if often, people consider equal two pens even if the colors of the pens are different. In algebra if 'your expression is equal to mine' we have written two identical expressions, for instance 2+3 and 2+3"*

Protocol 17 Danielas's (class 2003, grade 9) answers to the first four questions of the final test (see appendix **13.8.**).

The answers given by Elisabetta (Protocol 16) to the first four questions of the test gives a clear picture of process initiated by the teacher introducing the symbol "= =" to represent equivalence as opposed to equality. Pupils are asked to write what they think about the meanings of the symbol "=", and the words "equal" and "equivalent" in algebra, then they are asked to write what relationships they think there are among them, and if they know other related symbols (questions 1-4 of the test reported in appendix **13.8.** . For what concerns the symbol "=" and the word "equal" (questions 1 and 2), Elisabetta writes:

> *"From the beginning of this school year, till now, with our teach and my classmates, we established that the symbol '=' means 'EQUAL'. For instance if are given the expressions '5+3-2,7' and '5+3-2,7' we can write '5+3-2,7=5+3-2,7' because they are equal. For us, the word 'EQUAL' means 'identical', thus, even if, for instance, two expressions give the same result, but initially are not 'identical', we cannot state that they are equal. For instance: '2,3+2,5' and '2(3+5)' give the same result (16), but are not identical."*

In this excerpt Elisabetta identifies quite clearly the meaning she attributes to the symbol "=" and the word "equal" in algebra: two expressions are equal if they are identical, if they look exactly the same, and if that is the case it is possible to write one next to the other separated by the symbol "=". The use she does of the symbol "=" is clearly structural, in the sense that it is not used with a preferential direction, and is not interpreted as a separator between two computational steps; instead the symbol it is used in terms of comparison of expressions which thus are interpreted as object that can be compared and not necessarily have to be computed. Similarly her class mate Daniela (Protocol 17) writes:

"a) I think in algebra the symbol '=' has the meaning of equality, that is, it indicates that two elements that it compares are equal, identical. […]"

In her answer, Elisabetta, gives an example of two expressions that are not identical, thus not equal, but they have the same numerical result, with this example she prepares the territory for her answer to the following question (question 3), concerning the word "equivalent":

"Still basing on what I have learnt in these months, the word 'equivalent' means 'having the same result'. For instance if I consider the 2 expressions: '2˛3+2˛5' and '2(3+5)', even if they are not identical, I can notice that they have the same final result, or I can notice that if I apply the AXIOM OF THE DISTRIBUTIVE PROPERTY I can render them 'EQUAL'; thus they are 'equivalent' because they have the same final result or because by applying an axiom I can 'transform them' untill they are 'identical'. Sure enough, in the case of literal expressions which are not initially equal, I can prove their equivalence only by trying to render them equal, and thus equivalent."

The word "equivalent", for Elisabetta, has its autonomous meaning, based either on numerical computations or on axiom based transformations, both in the cases of numerical or literal expressions (in the latest only axiom based transformations work). The relationship between the word "equal" and "equivalent" is related to the fact that the proof of the equivalence of two expressions end when they are transfromed into two equal expressions. Equality is a means for proving equivalence, and it functions by constituting both the aim and the stop condition of the transformation process. It is curious how Elisabetta, as most of her class mates, uses the word 'identical' to explain equality, probably this is done in order to avoid the confusion brought by the polysemy of the word "equal" derived also from contexts external to the mathematical one. For instance, Daniela writes:

"a) I think that in algebra "equal" means identical. I think that an element is equal to another if they are identical.

b) This word is used in daily life. For instance 'your pen is equal to mine' and the two pens are identical, even if often, people consider equal two pens even if the colors of the pens are different. In algebra if 'your expression is equal to mine' we have written two identical expressions, for instance 2+3 and 2+3"

This example shows a great control of the different meanings associated to the word "equal", Daniela is able to distinguish different meanings according to different context, and offers an illuminating example of statements, one from real life context, and one from algebra context, that are obtaind one from the other just by substituting the word "pen" with the word "expression": the statements are almost identical, but the different context changes the meaning of the word "equal", as described by Daniela.

A similar mastery of the polysemy of the considered words and symbols of equality and equivalence, is showed by Elisabetta's answer to the fourth question. Elisabbetta, looking for other related symbols, introduces the symbol "= ="[47]:

"In our class, the symbol '=' is used to indicate that two expressions are "equal", while the symbol "= =" is used to indicate that two expressions are equivalent. Of course when I succed, by means of the application of theorems and axioms, to transform an expression into an equivalent one, at the end I obtain two expressions that apart from being equivalent, they are also equal, and in this case both the symbols "=" and "==" can be used."

[47] Notice that in the first part of the test there was no reference to L'Algebrista, which was mentioned in the second part that was submitted to pupils only after they had finished the first.

Elisabetta explains the distinction between the symbol "=" and the symbol "= =", similarly to the case of the words "equal" and "equivalent".

These two protocols can be considered exemplar of the answers that can be obtained, at this moment, from the pupils. The distinction between "=" and "= =" and between "equal" and "equivalent", which was on purpose introduced by the teacher by means of L'Algebrista, is finally closed. Pupil seem to be able to handle the polysemy of the sign "=", a polysemy which reached a stability in terms of the oppositions "equal"/"equivalent" and "="/"= =", and which is based on the sphere of practice of proving equivalencies of expressions in the microworld.

9.3. The idea of theory

According to our theoretica framewrok, one the basic characteristics of mathematics is considered to be its theoretical organization in terms of axioms, definitions and theorems:

> “the theoretical organisation according to axioms, definitions and theorems, represents one of the basic elements characterising mathematical knowledge”
>
> (Mariotti, as cited in **[3]**, Balacheff, 2002)

This theoretical organization defines the critaeria of acceptabilty of a theorem in terms of its validity in a theory, independently from any empirical verification:

> “A theoretical fact, a theorem […] is acceptable only because it is systematised within a theory, with a complete autonomy from any verification or argumentation at an empirical level”
>
> (Mariotti, as cited in **[3]**, Balacheff, 2002)

Drawing from these considerations, the key principles of our educational strategy are:

- We do not considers "generic" theories, but theories shared by a community, in particular, we may speak, for example, of mathematicians' theories, and of theories of the classroom.
- The axioms and definitions constituing the considered theory have to be clearly stated and distinguished from other statements
- A theorem makes sense only with respect to a theory a theory, within which a proof is provided
- Given a statement, in order to derive from it a theorem of a theory, the validity of the statement has to be proved by means of the elements of the theory
- For a valid statement to become a theorem of a theory, it has to be shared and accepted as such by the community itself, i.e. in school practice it has to be shared and accepted by the class.

These principles individuate also some of our educational sub-goals in relation to the aim of introducing pupils to theoretical thinking. On the basis of these principles we are going to discuss some results obtained in our experimentation.

9.3.1. Creating new theorems with L'Algebrista

In the first phase of our teaching sequence pupils are involved in activities of transforming numerical expressions by means of the stated properties, at the same time, thruogh collecttive discussions this practice has been related to the algebraic activity of proving equivalencies of numerical expressions by means of axioms. in this way for this reason, when letters are introduced, pupils are already familiar with the practice of using axioms as means for proving particular statements. Such a practice is then extended to the case of literal expressions, introducing activities of proving of statements of equivalence of literal expressions.

*9.3.1.1.**The first theorem***

The first activity with literal expressions that we propose to pupils, requires pupils to conjecture which of a set of given literal expressions can be interpreted as correct formulas for computing the area of a given geometrical figure (see activity **CL 7** in appendix **13.3.**). This activity involves comparison of literal expressions, and we observed that pupils tend to employ the comparison techniques they had learnth with numerical numbers. Namely, they tried to prove the equivalence of literal expressions by means of the axioms they had been using with numbers, as in the case of Eleonora (Classe 1998, **Protocol 9**) that we discussed in a previous section (**9.2.3.2**). However, at that point, the transformation principles pupils could use were a limited number, as showed by the corresponding buttons available in the "Teoria" (set of commands corresponding to the axioms of atheory) of L'Algebrista that they were using (Figure 21).

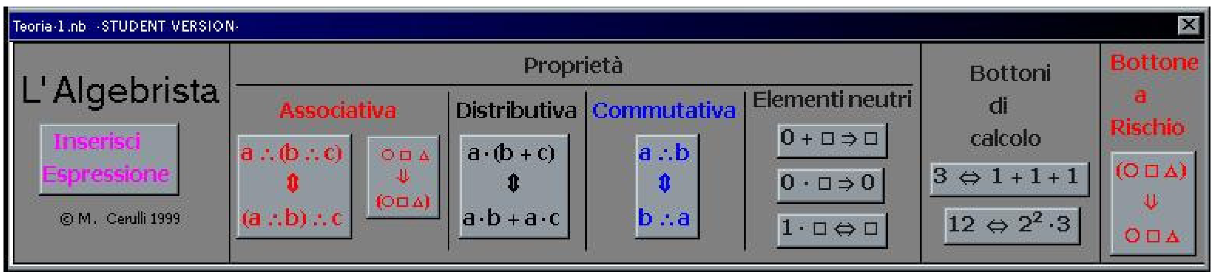
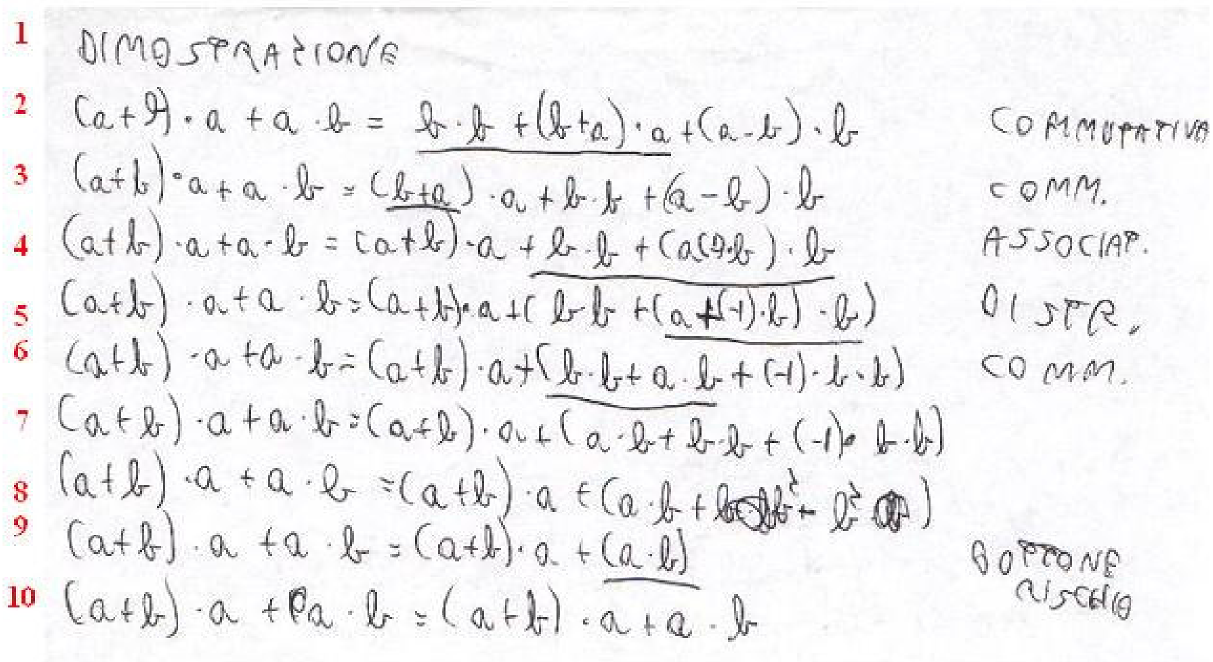


Figure 21 At the moment of their first activity with literal expressions, the class theory includes only a few numbers of axioms; in fact, pupils can use only the buttons of "Teoria 1" of L'Algebrista.

In particular, no rules for summing monomials, and for managing powers, were formalised and shared by the class as elements of their official theory, and in fact there were no corresponding buttons in L'Algebrista; nevertheless, the proposed activity was designed in order to have pupils facing the need of using such rules as means for proving the equivalence of the considered expressions. As a consequence we had behaviours like that of Tiziano (classe 1999, grade 9, Protocol 18) who behaved in the same way as that Eleonora (Classe 1998, grade 9, **Protocol 9**) that we previously described.



1 DIMOSTRAZIONE

2 $(a+b)\cdot a+a\cdot b=b\cdot b+(b+a)\cdot a+(a-b)\cdot b$ COMMUTATIVA

3 $(a+b)\cdot a+a\cdot b=(b+a)\cdot a+b\cdot b+(a-b)\cdot b$ COMM.

4 $(a+b)\cdot a+a\cdot b=(a+b)\cdot a+b\cdot b+(a(-1)\cdot b)\cdot b$ ASSOCIAT.

5 $(a+b)\cdot a+a\cdot b=(a+b)\cdot a+(b\cdot b+(a+(-1)\cdot b)\cdot b)$ DISTR.

6 $(a+b)\cdot a+a\cdot b=(a+b)\cdot a+(b\cdot b+a\cdot b+(-1)\cdot b\cdot b)$ COMM.

7 $(a+b)\cdot a+a\cdot b=(a+b)\cdot a+(a\cdot b+b\cdot b+(-1)\cdot b\cdot b)$

8 $(a+b)\cdot a+a\cdot b=(a+b)\cdot a+(a\cdot b+b^2-b^2)$

9 $(a+b)\cdot a+a\cdot b=(a+b)\cdot a+(a\cdot b)$ BOTTONE RISCHIO

10 $(a+b)\cdot a+a\cdot b=(a+b)\cdot a+a\cdot b$

Protocol 18 Tiziano (class 1999, 9th grade)explains most of the steps of his proof by means of the axioms of the theory officially shared by his class. However, the transformations from step line 7 to line 8 and from line 8 to line 9, do not correspond to any of such axiom but are needed in order to tranform the rightmost expression into the leftmost one. As a consecuence Tiziano executes such transformations, but fails (or avoid) explaining them in terms of the class theory. The steps of the protocol have been numbered by the author of this dissertation.

Tiziano, explains each step (exluded two steps) of his proof by means of the axioms of the theory officially shared by his class; in fact for each step, he writes (right column of the protocol) the used axiom, or corresponding button of L'Algebrista, and underlines the sub expression on which the axiom is applied. However, the transformations from line 7 to line 8 and from line 8 to line 9, are not obtained directly by means of the axioms shared by the class at that moment (see Figure 21), in fact, Tiziano executes them without explaining them in terms of the theory of the class. The first of these transformations corresponds to the definition of power, in fact Tiziano (lines 7, 8) transforms **b•b** into $\mathbf{b^2}$, while the second one (lines 8, 9) corresponds to the "regola di cancellazione" (en.: "cancellation rule") by means of which the pupil deletes the sub expression $\mathbf{b^2\text{-}b^2}$. a capo.

It is interesting to observe that from line 7 to line 8, Tiziano partially abandons the notation of L'Algebrista, transforming the sub expression **b•b+(-1)•b•b** into the sub expression $\mathbf{b^2\text{-}b^2}$. The new sub expresssion contains the difference of the terms $\mathbf{b^2}$ and $\mathbf{b^2}$ which in L'Algebrista would be represented as $\mathbf{b^2+(-1)\bullet b^2}$. The notation used by Tiziano here is that of standard computations in paper and pencil, and in fact, also the transformations he applies are derived from that practice. Tiziano seems to be conscius of the difference of the two practices, as indicated by the fact that he changes notation, and by the fact that he doesn't apply the *cancellation rule* directly to the sub expression **b•b+(-1)•b•b**, transforming it into $\mathbf{b^2\text{-}b^2}$ before applying such rule. In fact the *cancellation rule*, in standard calculation of numerical expressions, is used to delete from an expression a sub expression made of the *difference of two equal numbers*. In other words Tiziano transforms the sub expression **b•b+(-1)•b•b** into a form, $\mathbf{b^2\text{-}b^2}$, on which he feels he can apply the cancellation rule.

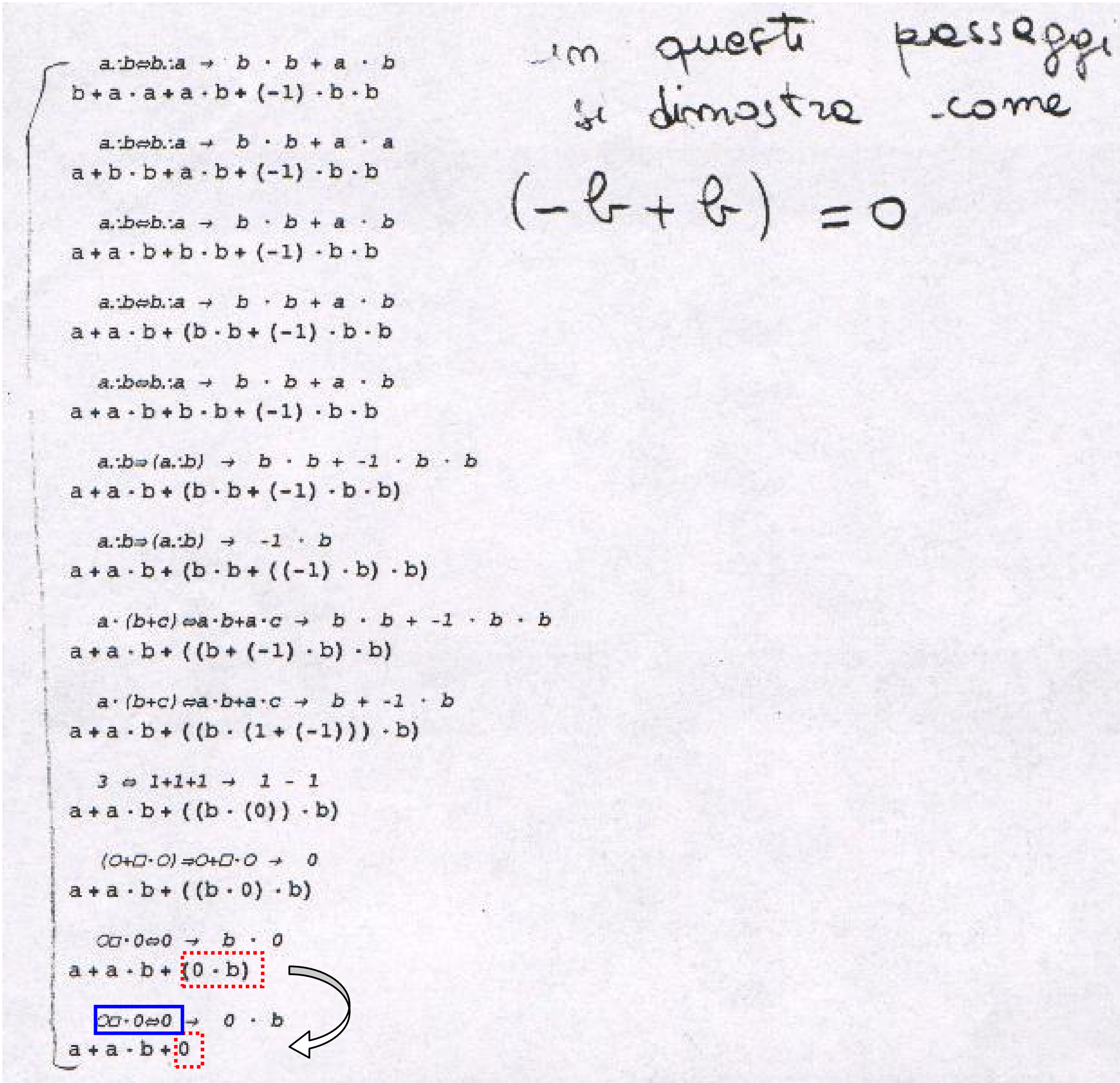
In the case of numerical expressions, in the case of standard computations, a product of two equal numbers, for instance **3,3** would simply be computed obtaining **9**, but this is impossible with letters; however, pupils seem to follow a need of simplification, and try to simplify, an expression such as **b•b,** as much as possible by transforming it into $\mathbf{b^2}$ which seems not to contain operations to be executed. Equivalently, the difference of two numbers, for instance **3-3**, in standard numerical computations, would simply be transformed into **0** or cancelled. In the case of literal expressions, pupils try to behave similarly and simply delete **b-b** or transform it into **0**.

In the case of the activity that we are commenting on, Tiziano needs to eliminate the sub expression **b•b+(-1)•b•b** from the rightmost expression of line 7, **(a+b)•a+(a•b+b•b+(-1)•b•b)**, because he wants to transform it into the leftmost expression, **(a+b)•a+a•b**. But none of the axioms shared by the class at that moment allow such deletion within a single step, and in L'Algebrista, of course, there is no button that simply deletes the sum of two opposite literal expressions such as **b•b** and **(-1)•b•b**. As a consequence Tiziano, can only abandons temporarily the practice of *proving* by means of axiom or corresponding buttons, and execute the transformation steps of line 7, 8, and 9, by using transformation rules that are external to the practice of *proving*. In this case L'Algebrista functions as an intrument of semiotic mediation in the sense that it represents the practice of proving by means of axioms, and Tiziano simply stops using signs derived from L'Algebrista in the moment that he changes kind of practice. The microworld, used by Tiziano both with its presence and its absence, functions as an external control individuating the practice of *proving* by means of axioms as opposed to the practice of standard calculation.

The transformations that Tiziano (like Eleonora, **Protocol 1**, and most of the pupils of the experimentation) executes without giving explanations are that of cancelling two opposite terms and that of writing the product **b•b** as $\mathbf{b^2}$. Such behaviour was possible in the paper and pencil enviroment, but it was not in L'Algebrista because at that point, in the microworld, pupils didn't have any button corresponding to such transformations among the available buttons. However, the "proof" produced by Tiziano, and other similar ones, cannot be accepted by the class community as a valid proof in their shared algebra theory, because it isn't expressed in terms of the axioms of the available theory, at the moment represented by the palette of L'Algebrista. for this reason, the teacher, on purpose, suggests[48] pupils to execute their proof in L'Algebrista where the available buttons correspond only to the shared axioms, and it is not possible to execute transformations that cannot be explained by means of the axiom of the theory shared by the class. The microworl is thus used by the teacher as an instrument of semiotic mediation in the sense that it is used as a means to keep separeted the practices of *proving* from the practices derived from standard calculations, and it is used to provide pupils with an external control on the transformations they perform.

The case of the cancellation of opposite terms is particularly interesting, even if there is no button corresponding to it, it can be executed by applying a sequence of other tranformational steps based on the available buttons. How this can be done is shown in the chain of transformation produced by Sandra and Lucia (class 1998, grade 9), Protocol 19.

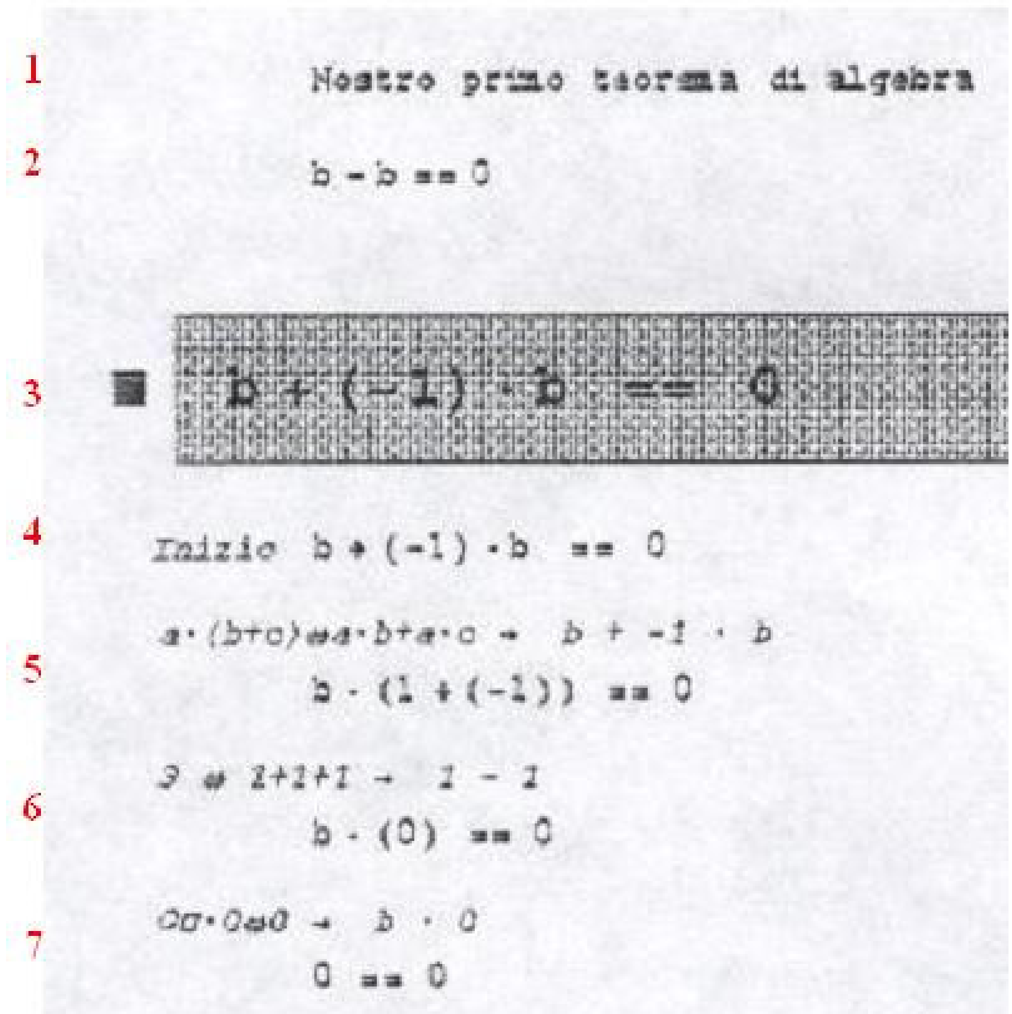
[48] This is done systematically in all the experiment we conduce.



Protocol 19 This is an excerpt of a longer chain of transformations (produced with L'Algebrista and printed on paper by the pupils); In this excerpt Sandra and Lucia individuate a proof of the statement "**–b+b=0**". The two girls write a comment: "In this steps it is proven how (–**b+b)=0**". Each new transformation step is obtained by selecting a sub expression from the last obtained expression, and clicking on a button of those available in the microworld. In the last lines we added some marks in order to help the reader understand the protocols; each transformation step is made of two lines, a smaller one on the top, and a bigger one on the bottom, the first contains the formula representing the used button, while the second represents the new obtained expression. In the last steps, the big dotted, red, rectangle indicates what sub expression was selected, the blue rectangle indicates the used button, and the last small red dotted rectangle represents the obtained sub expression. In this case the button used tranforms a product of an expression by zero, into zero. Following such a scheme it is possible to read and intterpret all the other steps.

Sandra and Lucia, executes a complex chain of transformations (an excerpt is presented in Protocol 19), which involves also transforming **b+(-1)•b** into **0**. The two pupils then printed on paper their chain of transformations and commented it highlighting (by marking them with a left bracket) the steps that are needed to transform **b+(-1)•b** into **0**. Such a sequence of tranformations can be interpreted as a proof of the fact that **b+(-1)•b** is equivalent to **0**; then, by interpreting the letter **b** as "any expression" (which implies "any number" as a number is considered to be also an expression in our experiment), it is possible to interpret this chain of transformation as a proof of the principle stating that the sum of two opposite terms can be substituted by zero. As a consequence, a corresponding theorem, the first one produced by the class, can be introduced in the shared theory .

In all the experimentations pupils behaved basically in the same ways with respect to this activity, which was always followed by a class discussion in which the new transfomation rule was socialized and accepted as the first theorem. The word "Teorema" (en.: "Theorem"), is on purpose introduced by the teacher, as a way to give a particular status to this statement which raised from a contrast between the paper and pencil environment and L'Algebrista, and which was proved by means of the stated axioms. However, prior to accept this statement as a theorem, the teacher requires the class to produce a proof that can be shared by the community and taken as their official proof; also this last step is guided on purpose by the teacher aiming at establishing socially shared rules for the acceptance of a theorem within the community. In Protocol 20 we show an example of proof produced by Marco (class 1999, grade 9) with L'Algebrista.

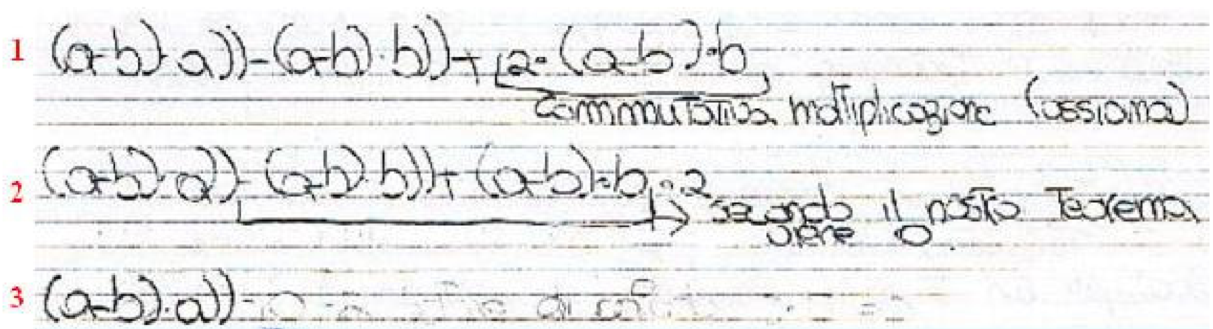


Protocol 20 The proof of the class' first theorem, made by Marco (class 1999, grade 9) in L'Algebrista. Marco first writes "Our first theorem" (line 1), then writes **b-b==0** (line 2) and inserts it in the microworld using the button "Insert expression". The entrance in the microworld is marked by a title line (line 3) and a starting line (line 4) marked by the lable "*Inizio*" (en.: "start") and containing the expression **b+(-1) •b==0** on which Marco applied the distributive property obtaining the expression of line 5. Line 6 is obtained from line 5 by selecting **1+(-1)** and applying the computation button "3<=>1+1+1" which executed the sum of the two numbers. Finally, line 7 is obtained from line 6 by selecting **b•0** and clicking on the button "ç¸0<=>0" which transforms the product of zero by any expression into zero. The last line (7) presents the identity **0==0**, a stop condition for the transformation process which allows to state that the initially questioned equivalence **b-b==0** actually holds.

At this point, a new kind of transformation rule is introduced in the class theory, to which pupils oftern refer as to "our first theorem". Being a new element of the theory, the theorem corresponding to the new transformation rule, is added to the the *class algebra notebook* which already contains the previously known transformation rules. This new element is different from the previous ones because they are axioms, while this one is a theroem, such distinction is reported in the notebook by calling it *theorem* (as opposed to axioms and definitions) and by associating the theorem with its proof. In parallel with the edition of the notebook, a new button is added to the commands available in L'Algebrista; this operation is done by the pupils themselves, using "il Teorematore" (en: "The Theorem maker").

Figure 22 The button added to L'Algebrista by pupils after proving their first theorem.

The insertion of a new button, corresponding to the proven theorem, in L'Algebrista, results in the possibility to use the button to prove equivalencies of expressions or to prove theorems, in the same way as previously available buttons were used. As a consequence, the proven theorem, becomes a new instrument for proving statements, in the same way as axioms can be used to accomplish such tasks; for instance Marta (class 1998, grade 9) uses this new theorem, calling it "nostro teorema" (en.: "our theorem"), as shown in Protocol 21.



Protocol 21 Marta, required to prove the equivalence of the expression of line 1 and the expression of line 3, uses the first theorem use proven and accepted by the class as a mean to transform an expression. She calls the theorem "nostro teorema" (en.: "our theorem"), as she reported in line 2, when she writes "according to our theorem this becomes 0". The pupil is also required to indicate wither she uses a theorem or an axiom, that why in line 1 she writes "commutativity of multiplication (axiom)"

*9.3.1.2.**Buttons in L'Algebrista and theorems in the mathematical notebook***

After the introductory activity with literal expressions, pupils are proposed activities of proving equivalencies of literal expressions, some of which are chosen for being included in the *class algebra notebook* as new theorems, and are at the same time added to the microworld in the form of new buttons. As a consequence, the class algebra notebook, and the set of buttons available in L'Algebrista, grows up in parallel together with the algebraic knowledge shared by the class. Each pupil has its class algebra notebook, and the coherence among the single/individual notebooks is obtained by means of periodical class revisions of the notebook, and by social sharing practices guided by the teacher.

A skectch of the possible evolution of the class algebraic knowledge can be found in the class algebra notebook of Marco, a pupil of class 2000.

TEORIA 0

ASSOCIATIVA MOLTIPLICAZIONE E ADDIZIONE

$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

DISTRIBUTIVA DELLA MOLTIPLICAZIONE RISPETTO ALL DIZIONE

$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

COMMUTATIVA ADDIZIONE E MOLTIPLICAZIONE

$a + b = b + a$

BOTTONE A RISCHIO (NON É UN ASSIOMA)

$((a+b) \cdot c) = (a+b) \cdot c$ CORRETTO

$((a+b) \cdot c) = (a+b \cdot c)$ SCORRETTO

TEORIA 1

ASSOCIATIVA (METTI PARENTESI)

$a + \underline{b \cdot c} = a + (b \cdot c)$ CORRETTO

$\underline{a + b} \cdot c = (a+b) \cdot c$ SCORRETTO

ELEMENTI NEUTRI

$0 + \square = \square$

$0 \cdot \square = 0$

$1 \cdot \square = \square$

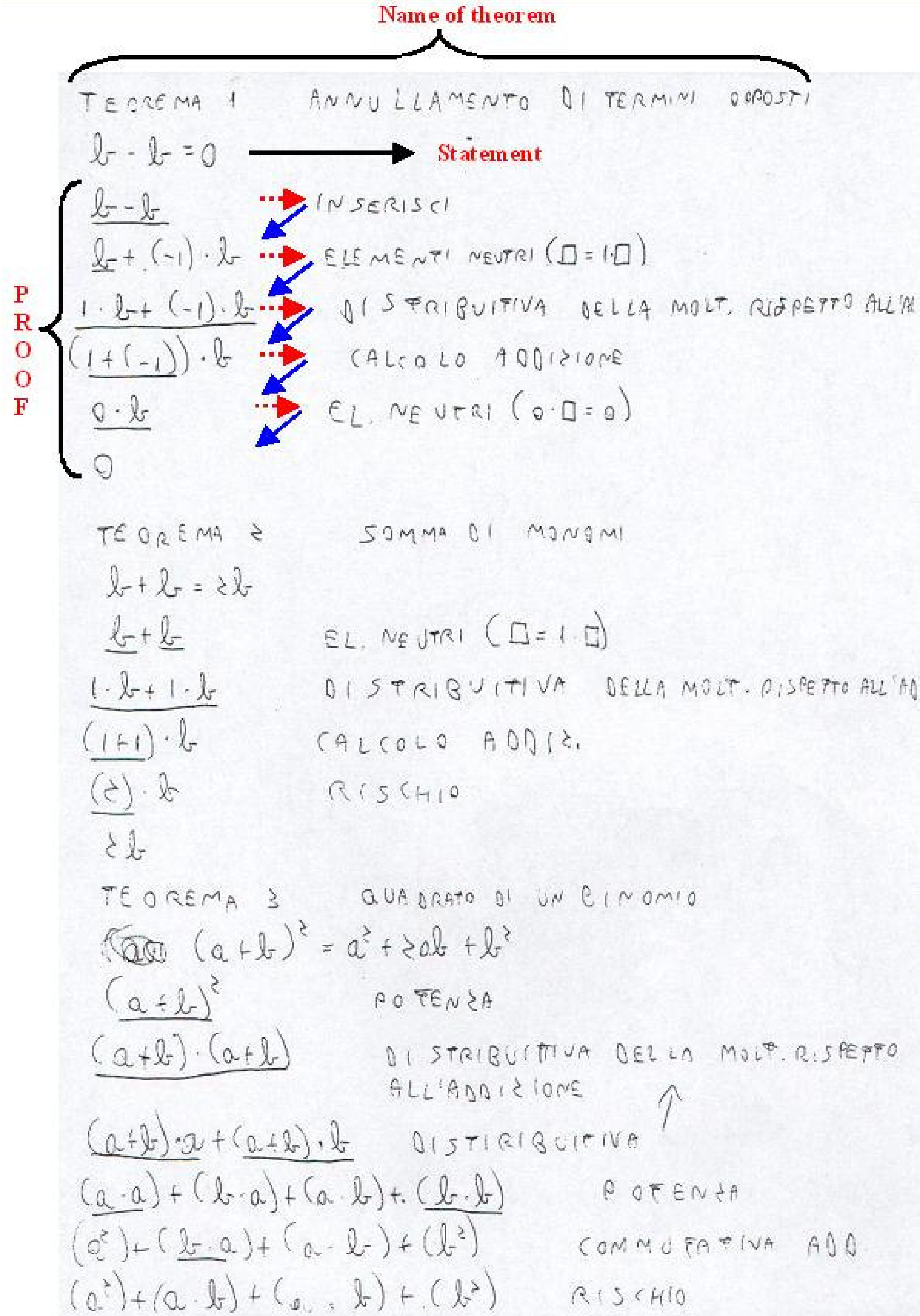
TEORIA 2

DEFINIZIONE DI POTENZA

$a^6 = a \cdot a \cdot a \cdot a \cdot a \cdot a$

$b \cdot b \cdot b = b^3$

Protocol 22 Marco's class algebra notebook. Firstly it contained only the axioms of "Teoria 0" the first theory used by the class with L'Algebrista; then other axioms were added, those of "Teoria 1" and "Teoria 2".


Name of theorem

TEOREMA 1 ANNULLAMENTO DI TERMINI OPPOSTI

$b - b = 0$ Statement

PROOF

$b - b$ INSERISCI

$b + (-1) \cdot b$ ELEMENTI NEUTRI ($\square = 1 \cdot \square$)

$1 \cdot b + (-1) \cdot b$ DISTRIBUTIVA DELLA MOLT. RISPETTO ALL'AD

$(1 + (-1)) \cdot b$ CALCOLO ADDIZIONE

$0 \cdot b$ EL. NEUTRI ($0 \cdot \square = 0$)

0

TEOREMA 2 SOMMA DI MONOMI

$b + b = 2b$

$b + b$ EL. NEUTRI ($\square = 1 \cdot \square$)

$1 \cdot b + 1 \cdot b$ DISTRIBUTIVA DELLA MOLT. RISPETTO ALL'AD

$(1+1) \cdot b$ CALCOLO ADDIZ.

$(2) \cdot b$ RISCHIO

$2b$

TEOREMA 3 QUADRATO DI UN BINOMIO

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^2$ POTENZA

$(a+b) \cdot (a+b)$ DISTRIBUTIVA DELLA MOLT. RISPETTO ALL'ADDIZIONE

$(a+b) \cdot a + (a+b) \cdot b$ DISTRIBUTIVA

$(a \cdot a) + (b \cdot a) + (a \cdot b) + (b \cdot b)$ POTENZA

$(a^2) + (b \cdot a) + (a \cdot b) + (b^2)$ COMMUTATIVA ADD

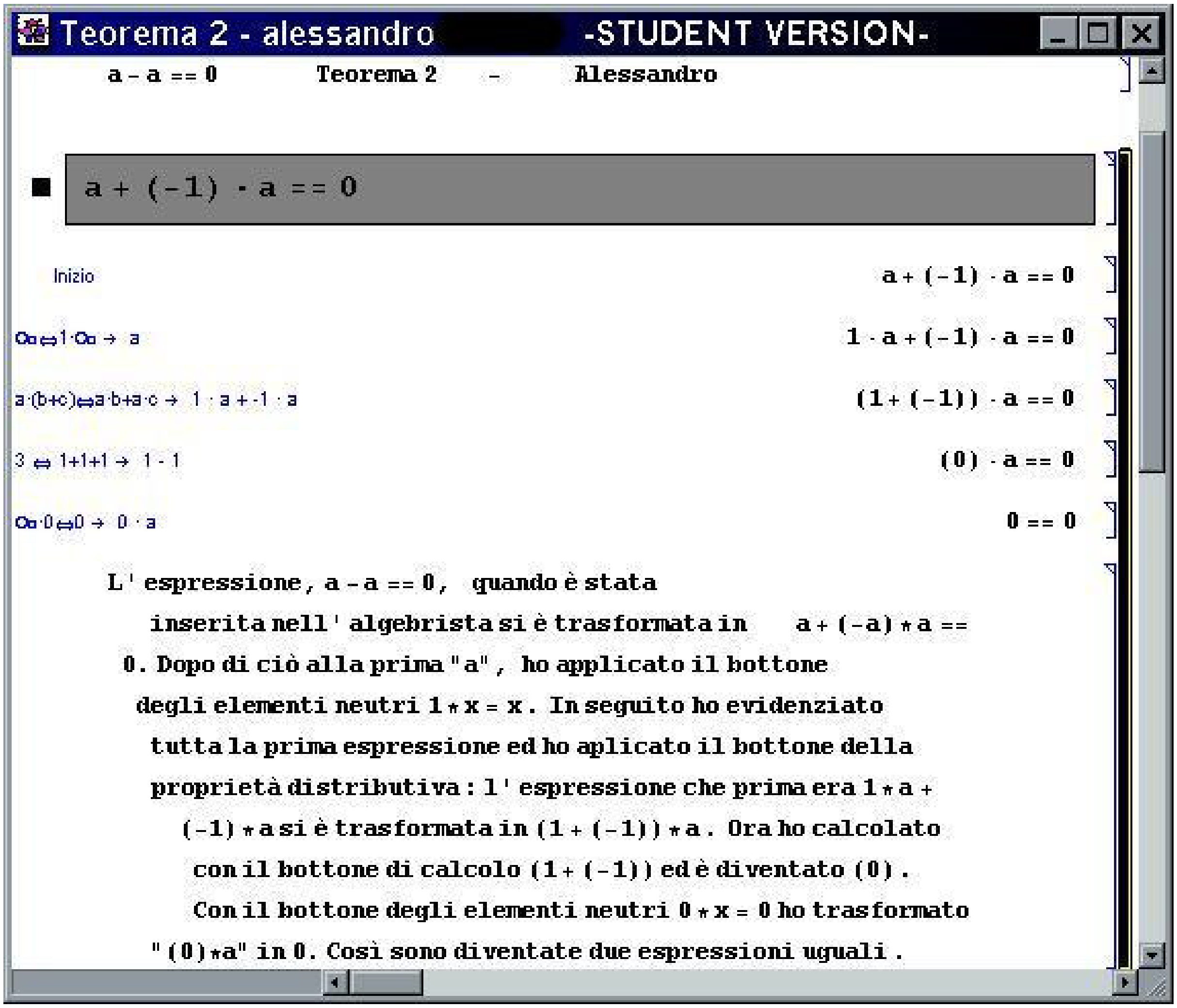
$(a^2) + (a \cdot b) + (a \cdot b) + (b^2)$ RISCHIO

Protocol 23 Also theorems are added to the class algebra notebook, when they are proved and shared by the class. Differently from the case of the axioms, in the notebook it is reported, together with the statement, also a proof of its validity; such proof is made by means of the elements already present in the theory represented by the class algebra notebook. We added some marks in order to help the reader understanding the structure of the notebook.

The subsequent enlargements of the theory are clearly shown: the notebook (see Protocol 22) firstly contained only the part "Teoria 0" with the first axioms used in class, which were derived from the first set of command used in L'Algebrista. As the class went forward, using other theories in the microworld, thus new buttons, and the corresponding axioms were added to the class algebra notebook, like the axioms of "Teoria 1" and "Teoria 2" that can be seen in Protocol 22. Also theorems are added to the class algebra notebook (see Marco's notebook in Protocol 23), when they are proved and shared by the class. Differently from the case of the axioms, together with the statement, in the class algebra notebook it is reported also a proof of its validity, so that proof becomes an integral part of what is recognized as a Theorem; each proof is made by means of the elements already present in the theory and represented in the class algebra notebook.

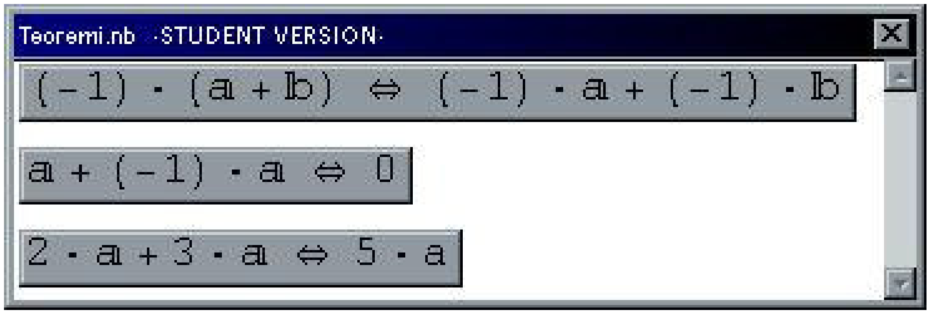
The reference to the activity carried out in the microworld although not explicit can be recognized in the editing of the chain of transformation .Observe that, in any of the proofs reported in Marco's notebook, the used axioms or theorems, are never reported next to the expression obtained using them: they are always reported next to expression on which they are applied. Axioms and theorems are not used to comment on produced transformation, but are used to produce transformations, exactly as it happens with L'Algebrista. In Protocol 23 we hilighted some lines in order to help the reader following the flow of one of the reported proof.

Within the sequence of activities that we propose, theorems are sometimes proved in paper and pencil, and sometimes in L'Algebrista. However, for each new theorem of the class theory, a new theorem is inserted in the *class algebra notebook*, and a new button is created in the microworld; thus, pupils also produce, in L'Algebrista, their own palettes containing buttons corresponding to theorems. In the latest experiment, with class 2003, pupils' *class algebra notebooks* consist of documents produced directly with the computer; in L'Algebrista pupils write the statement of the theorem and a name for it, then click on the "insert expression" button and prove the theorem directly in the microworld. Once they finish their proof, they print on paper the obtained document, and use it as a page to be inserted in their personal *class algebra notebook*. According to the teacher, pupils decided to edit the notebook directly on the computer because they realised that when you have to modify your notebook is easier if you have it in electronic format. A typical page of these new notebook is that of Alessandro where one can find the proof of the theorem that the sum of two opposite terms is **0** (see Protocol 24).



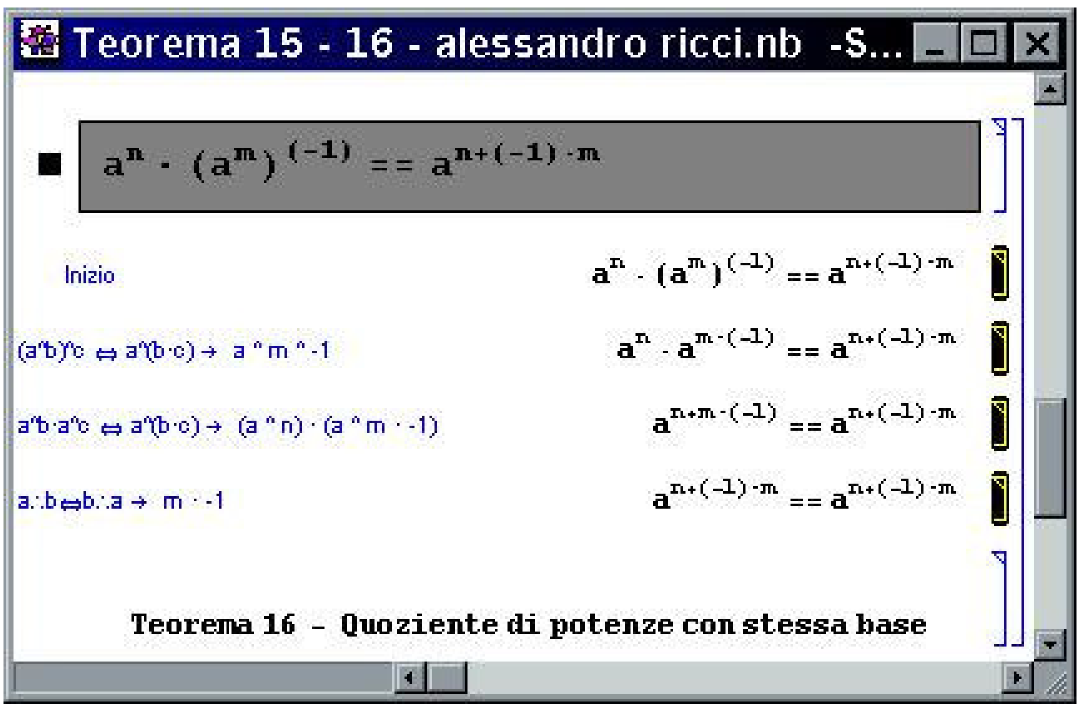
Protocol 24 This is Alessandro's proof of the theorem discussed in section. Th pupils, transforms the expressions by means of the axioms, and also explains verbally what he has done: "The expression, a-a==0, whene it was inserted in L'Algebrista, it became a+(-1)*a==0. After that, to the first "a", I applied the button of neutral elements "1*x=x". Then I selected the whole leftmost expression and I applied the button of the distributive property: the expressions that previously was 1*a+(-1)*a was transformed into (1+(-1))*a. Now I computed with the computation button (1+(-1)) which became (0). With the button of neutral elements 0*x=0 I transformed "(0)*a" in 0. So they became two equal expressions.

However, even if notebooks are in electronic format, they are not identified with L'Algebrista. Actions in the two context are easyly recognized as different. On the one hand, the enlargment of the algebra theory corresponds to the update of the notebook, which is accoplished by writing the new statement and its proof. Interesting to remark, that, in the case of Marco, although the proof is copied form the microworld, it is enriched by comments, that are non required in the microworld. on the other hand, the enlargemnt of the theory represented in the microworld can evolve only by means of insertion of new buttons. In fact, pupils, in parallel with the growth of their notebook, add new buttons to to L'Algebrista, creating their own palettes of the theorems. For instance, after proving the first three theorems, the personal palette of theorems of Alessandro, in the microworld, was the following one, which includes also the "Teorema 2" whose proof is presented in Protocol 24.



Protocol 25 The set of buttons created by Alessandro in L'Algebrista after proving the first three theorems, the second one correspond to the theorem proved in **Protocol 24**.

To conclude, we observe that along with the evolution of the theory, the proofs of the theorems inserted by pupils in the notebook, become more schematic, in the sense that they do not include verbal explanations anymore, as shown in Protocol 26, that is a document of Alessandro's notebook reporting a theorem concerning negative powers. This can be interpreted as a trace of the process of authomatization of the transformation process, as a step towards the construction of a theoretical meaning of “calcolo” the was one of our didactical aims .



Protocol 26 Alessandro proves a theorem concering negative powers.

9.4. An example of use of L'Algebrista as an instrument of semiotic mediation

In the previous sections of this chapter we presented some results, highlighting the potentialitis of L'Algebrista as an instrument of semiotic mediation. According to our hypothesis, such potentialities can be exploited by the teacher by means of particular communication strategies, some of which we are going to discuss in this section. We will draw from the analysys of the transcript of an episode that took place in the final phase of the experiment in class 2000.

9.4.1. Presentation of the episode

The episode is a collecitve discussion that takes place at the beginning of the second school year of experimentation with the grade 10 pupils of class 2000. During the previous year pupils have been following the experimentation for 9 months, and before summer holidays the teacher began to introduce equations starting from the problem of comparing literal expressions, to which pupils were familiar. The idea is that if we consider expressions that are not equivalent, then we may ask if it is possible to find any number such that if we substitute it to the letters, the obtained numerical expressions result to be equivalent. In the episode, some axioms and theorems to be used for solving equations have been already introduced and discussed. The following excerpts drawn from the transcript of the discussion, highligh the strategy used by the teacher and her key interventions that exploit L'Algebrista as an instrument of semiotic mediation. what make this episode particularly interesting is the fact that the functioning of the microworld, as an instruments of semiotic mediation, appears when the microwolrs and its tools are not directly available because. in fact, the episode, we are going to analyse, occurred when the class was not in the computer laboratory.

9.4.2. The axiom theorem

The teacher (T) begins the lesson by asking pupils to recall what they said 3 months earlier about equations. She is aware that in class the word "equation" have already been introduced, and the statement "**A=B <=> A–B=0**" was has already been discussed once in by the class. The main aim of this discussion is to formalise the statement "**A=B <=> A–B=0**" as a principle for solving equations ,, and next to introduce other principles to be used as means for solving equations.

Excerpt 1 (class 2000)

1. T: So, the first question is, do you remember what we have been doing at the end of last year? What did we focus on?
2. Tcl: The axiom theorem *(ita.: assioma teorema)*
3. Cri: axiom theorem one

   […]

6. T: What is it?
7. Tcl: if A is equivalent to B then A minus B is equivalent to zero.

The discussion is set up on the basis of the history of the class. It is interesting the intervention of Cri (3), it witnesses the class habit of naming theorems by numbering them, but also the fact that pupils are aware of the particular theoretical status of this statement, neither an axiom, nor a theorem. At this point in the *class algebra notebook* there are several numbered theorems, and several axioms, but there is only one "axiom theorem", which is perceived as different both from axioms and theorems, and for this reason is given the number one. The ambiguous status of such statement (axiom theorem) is rooted in the way the principle was socialised and shared by the class, with respect to their idea of theory.

Because an axiom theorem is not an axiom neither a theorem, the teacher asks pupils to recall the nature of such a new mathematical object.

Excerpt 2 (class 2000)

8. T: come here and write it *(on the blackboard),* then explain why we called it axiom theorem

   […]

Tcl writes on the blackboard: a = = b <=> a – b

[…]

14. T: do you remember why did we call it axiom theorem? Is it normal to call something axiom theorem?

    […]

20. Bzc: we didn't know if…it was proved, we took it as an axiom, last year, but if later we are able to prove it … we left it undecided.

The axiom theorem is thus a statement whose status in the theory has been left undecided. On purpose, the class postponed a definitive decision, giving it the status of an object which is neither an axiom, nor a theorem, but that like axioms and theorems can be used in algebraic activities.

Excerpt 3 (class 2000)

21. T: […] So, we have this axiom theorem, what did it bring to us? What did we do with this axiom theorem? Nothing? Did we just look at it?

Once the "axiom theorem" principle is stated the teacher shifts the focus on its operative aspects ([21]). this point is coherent with the general approach of conceiving axioms and theorems also as instruments to transform expressions, but also with the general practice of using the corresponding buttons. Axioms and theorems are considered as corresponding to the buttons of L'Algebrista, and are used in proving practices, either in paper and pencil or in the microworld.

The teacher's intervention ([21]), starts a discussion in order to recall the formulation of the problem of solving an equation, in terms of comparison of expressions. Then the teacher proposes a new problem, taken from their holyday homeworks, and writes on the blackboard.

Excerpt 4 (class 2000)

89. Mrct: so, $a^3+b^3= =(a+b)(a^2-ab+b^2)$ *(reads the text of the equation to be solved)*
90. T: so, let's see… in the same stream of what we have been saying now, if we have two expressions and ask ourselves of which kind they are, whether they are equivalent or whether they are equal equal…if instead they give the same result only for some values of the letters or never. So, What did we use to start?
91. Stf: the check
92. T: That is, what did we use to do?
93. Stf: we used to substitue letters with numbers and then compute

[…]

Tcl executes the computation on the blackboard

$3^3+2^3= =(3+2)(3^2-2\bullet3+2^2)$ a=3 b=2

$27+8= =5\bullet(9-6+4)$

$35 = =5\bullet7$

$35 = = 35$

100. T: what can we conclude?
102. Mrs & Cri: we conclude that it is not impossible [that they are equivalent]

[…]

The first step to solve an equation is thus that of checking if the two expressions can be equivalent or not, this is done by substituting numbers to the letter, as shown in the excerpt. Here the symbol "= =" is used to represent a pending equivalence relationship, whilst the symbol "=" is used to represent assignment of numbers to letters. The use of different symbols for these two meanings, keeps them separated, avoiding confusion between equivalence of expressions, and the principle of substitution. The presence of the simbol “= =”, is quite normal in the paper and pencil context and the computation seem to take place without any reference to L’Algebrista.

At this point of the episode the class conjectures that the two expressions can be equivalent, thus they decide to prove it.

9.4.3. The "Insert Expression" button

In the first part of the discussion the teacher does not explicitly refer to L'Algebrista which is not available, as the episode takes place in a normal classroom with no computers. However, one of the aims of the teacher is that of introducing the axiom theorem in a new practice which will be that of solving equations. In our approach the operative aspects of axioms and theorems is fostered through the use of L'Algebrista, thus, in order to foster the new practice and the idea that the axiom theorem can become a new instrument for solving problems, the teacher decide to use the microworld as intrument of semiotic mediation and does her first (within this discussion) explicit reference to L'Algebrista.

Excerpt 4 (class 2000)

158. T: so, before using the axiom theorem, and before using the distributive property, we shell use "insert expression"

The button "Insert Expression" (*ita.*: "inserisci espressione") is the first command to be used in L'Algebrista in order to manipulate an expression: by clicking such button the expression is inserted into an environment where it is possible to transform expressions, and where it is possible to operate only using the available buttons. We recall that in this episode the microworld is not available, so the teacher does not aim at pupils to really introduce the expression in the microworld, but she just wants them *to act as if* they were working in L'Algebrista. This kind of intervention is quite usual in our experimentation, and at this point of the teaching sequence, which is almost at the end, pupils are familiar with this kind of requests.

We hypothesise that with this intervention the teacher has at least two specific aims:

27. when the teacher asks pupils to use the tool "insert expression" in paper and pencil, it is a kind of request of simulating the behaviour of L'Algebrista which in particular is rigorous and rigid; thus this intervention of the teacher may be interpreted as a request of respecting formal rigor.
28. the role played by buttons when using L'Algebrista is that of instruments to transform expressions, concrete instruments, thus the act of simulating the introduction of a new theorem (the "axiom theorem" here) into such kind of activities may foster its operative meanings.

Thanks to this intervention of the teacher, the symbols written on the blackboard become polysemic, in the sense that they refer both to mathematical meanings and to meanings derived from the practice in the microworld. This polysemy is exploited by the teacher recalling, when needed, the meanings related to L'Algebrista. For instance, consider the following example: Tcl has difficulties in applying the definition of power. The pupil begins his proof and gets stuck after two steps.

Excerpt 5 (class 2000)

Tcl executes the computation on the blackboard

$a^3+b^3==\underline{a(a^2-ab+b^2)}+\underline{b(a^2-ab+b^2)}$ Distributiva

$a^3+b^3==a\underline{a^2}+a(-1)ab+ab^2+ba^2+b(-1)ab+b\underline{b^2}$

194. T: what would you do here?
195. Tcl: it is **a a**
196. T: how do we do that...*(unanderstandable part)*…do you remember the symbol?
197. Tcl: I remember…I don't remember if…*(unanderstandable part)*
198. T: because…what is the button that does this thing?
199. Tcl: *(unanderstandable part)*

200. T: what…write it with the little triangles
201. Tcl: *(unanderstandable part)*
202. T: how is it? Do you remember the figure of the button?
203. Tcl: triangle to the second power equals triangle times triangle
204. triangle to the second power, double arrow, triangle times triangle

Tcl executes the computation on the blackboard

$$a^3+b^3==a\underline{a^2}+a(-1)ab+ab^2+ba^2+b(-1)ab+b\underline{b^2} \quad \square^2 <=> \square\bullet\square$$

$$a^3+b^3==\underline{aaa}+a(-1)ab+ab^2+ba^2+b(-1)ab+\underline{bbb} \qquad \square^2 <=> \square\bullet\square$$

The pupil is stuck, he wants to substitute "$\mathbf{a^2}$" with "**a a**" (191-195), he doesn't remember the definition of power and how to apply it. The teacher thus, referring explicitly to L'Algebrista, asks the pupil to recall the corresponding button of L'Algebrista (198). Once recalled the button, the teacher, on purpose, asks the pupil to recall the "figure/icon" of the button (202), thus deriving a formula, which is also a symbolic representation of the functionality of the button. Once recalled and written the formula, Tcl is able to apply it and goes on with his transformation steps. The formula, thus, not only represents the button and its functionality, but it is also represents a the definition of power, an its polysemy is exploited to bring in algebra practices the instrumental aspects typical of the buttons of L'Algebrista.

We observe that, in order to establish and exploit a plysemy of oral and written symbols in relation to practices with L'Algebrista, it is needed that the class share a knowledge concerning the relationship between practices within the microworld and paper and pencil mathematical class practices. Such relationship is built, in our approach, by means of activities of comparison of the two practices. At this point of the experiment the pupils of class 2000 are already quite familiar with such relationship, and are able to identify either parallelism and differences between the two spheres of practices and the related knowledge. In the following we are going to see how such relationship, and the polysemy of some key words, can be exploited to innescate and develop a process of creation of a new theorem.

9.4.4. The production of new theorems

From [158] to [236] the focus is on the solution of a specific equation, and we have few references to L'Algebrista, all of them concerning very operative aspects, like in the case of the episode of Tcl that we just discussed. Anyway till now the problem to be solved is a mathematical one (an equation) and the environment where to solve it is typical of mathematics: paper and pencil. Here the role of L'Algebrista is basically to provide instruments to solve such problem, but what pupils are really using are paper and pencil versions of such instruments, they are neither using buttons, nor transforming expressions written on the screen of a computer. In the following part, even if the computer is still absent, there is a change of the focus of the discourse caused by teacher's intervention [237]:

Excerpt 6 (class 2000)

237. T: […] have we ever solved equations within Lalgebrista?
238. Corus: no
239. T: no. So, Michele *(the developer of the software)* is here, we want to tell him what buttons we need in order to solve equations. He will add buttons to L'Algebrista, so what buttons will we require him to add?
240. Fmn: the axiom theorem!

The axiom theorem has been produced in the context of algebra, with no reference to L'Algebrista. It is a statement whose status is not yet well defined, but that is anyway already used to solve the problem of finding the solutions of certain equations. Thus, there is a need of including the axiom theorem in L'Algebrista, in the form of a button produced with the "Teorematore" ("theorem maker", see). The "Teorematore" is evoked [239] by means of a reference to the author of the software, Michele. Such evokation rises the problem of adapting the formulation of the statement to the syntax accepted by L'Algebrista.

First of all the teacher tries to point out the fact that the syntax used by the pupils is not the same as the syntax used by L'Algebrista, where "**a-b**" would be written "**a+(-1)*b**", so she asks:

Excerpt 7 (class 2000)

272. T: ok? But…do you think L'Algebrista would like such button?
273. Cri: no
274. T: why?
275. *confusion*
276. T: if we think of L'Algebrista's mentality …
277. Cri: that from a minus b equal zero I get a equal b it is ok? …
278. Tcl: ah! Because we need to do the "insert expression"
279. T: that is … he (*L'Algebrista*) doesn't like so much that "a minus b"…

Tcl gets to what the teacher is aiming at, and recalls the "insert expression" [278] button that, as a class convention, brings with itself also the specific syntax of L'Algebrista. This witnesses another meaning associated to the phrase "insert expression", that of representing a particular syntax, which correspond to a particular functioning of the microworld. As a consequence the expression "insert expression", when working in the paper and pencil environment, can be used either to recall practices of L'Algebrista, either to recall its syntax, whilst in the microworld it stands for the beginning of transformational activities by inserting expressions in the working environmet of the microworld. Such a polysemy can be used by the teacher to direct the attention of the pupils toward any of these aspects, according to their needs. Here we may notice passage [276] where the teacher not only recalls L'Algebrista, but also its "mentality": there is a specific way of reasoning that is associated to the software, and the teacher tries to direct pupils toward such rationality. In fact Cri seems to be really reasoning as if she was L'Algebrista and feels uneasy:

Excerpt 8 (class 2000)

281. Cri: but, how do we know that, for instance, if a is equal to b then a minus b is equal to zero? If a is equal to b? I mean, in the other direction we can do it because a minus b equal zero then we can do a equal b…but…
282. T: so, did you get Cri's problem? She says "if I have that a minus b is equal to zero, then it is ok"…then she says "if a minus b is equal to zero it is ok to say that a is equivalent to b" but she doesn't agree that if a is equivalent to b then a minus b is equivalent to zero.
283. Cri: No: when you apply it …how can you know that a is equivalent to b?
284. T: and in the other direction how can I know that a minus b is equivalent to zero?
285. *silence*
286. T: so, Cri's problem is very serious, but we must clarify it because it seems not be so clear even for her, in fact she can see it only in one direction and no in the other one…so she says "I want this button from L'Algebrista", right? "But if I apply this button here" … she says "how can I, how can L'Algebrista know that a is equivalent to b so that it can transform it?"

[…]

288. T: how can we tell that to L'Algebrista?

Cri's problem is subtle, when we transform "A=B" into "A-B=0" we take that A and B are equivalent as hypothesis. But, unfortunately L'Alg. is not able to "assume hypotheses", as we can do, and it is impossible to "tell that to L'Algebrista". If L'Algebrista doesn't know such hypothesis, how can it apply the axiom theorem? In this case the button would work transforming mechanically "**A=B**" into "**A-B=0**", it would be responsibility of the user to discuss what may happen if A is not equivalent to B, in fact this cannot be automat zed in the microworld. It is interesting to observe that this discussion takes place without computers, and that the button corresponding to the axiom theorem doesn't exist yet, thus pupils are talking about an imaginary, hypotetical, button that they know it could be available soon.

To sum up what happened, the axiom theorem was produced by means of a class discussion, as related to algebra, and was experienced in such environment. After that, coherently to their practice of adding buttons to L'Algebrista in parallel with new axioms or theorems in their mathematical notebooks, the teacher suggested pupils the creation of a new corresponding button. In fact, coherently to what happened with other axioms and theorems encountered by pupils during the experimentation, the axiom theorem is aspected to be interpretable both as element of the class algebra theory and as element of L'Algebrista, and as such it should be usable in both contexts. In other words, pupils are faced the problem of defining a new instrument to be added in L'Algebrista as derived from the axiom theorem. However, the practice of L'Algebrista is peculiar, and differs in some crucial aspects form paper and pencil practices, this caused Cri's uneasiness.

As a consequence the teacher, who had previously planned to make this issue emerge, exploits Cri's intervention as a input to move the focus of the discussion out of the microworld, back to Algebra world, where the solution of equations will be treated. In fact, from now on the teacher goes back to talk of only axioms and theorems and does not speak of "buttons" any more, she does not explicitly refer to L'Algebrista any more.

Excerpt 9 (class 2000)

340. T: so, that's right, Cri found a problem and she said "but how can L'Algebrista apply such a button? Because" she says "I have the expressions, I write two expression with an uqual sign between them, maybe I invent them and I absolutely don't know if they are equivalent or not, then I ask him (*L'Algebrista*) to apply such button and he brings the second expression at the first member, he puts a plus minus one before it, and tells me that it is equivalent to zero. But this is not true in case the two given expressions are not equivalent; it is not true that this (a+$(-1)b$) is equivalent to zero". But what is the problem we are tackling, Cri? Now are we asking L'Algebrista to transform expressions into equivalent expressions, or are we asking L'Algebrista to solve equations?
341. Cri: solve equations
342. T: to solve equations. Thus, maybe I can ask, I mean my problem is to question for what values of the letter those two numerical expressions are equivalent, thus I will have some values of the letter for which the numerical expressions are equivalent, thus it is ok to do such a passage; and I will have some values for which I do not know. Now, the important thing becomes another one, if they are not (*equivalent)*, how will their difference be?
343. Cri: different from zero
344. T: right, it will be different from zero
345. T: but also in that case, we shell take this as an axiom otherwise what do we have? A

monster?

346. T: so, delete everything *(from the blackboard)*, keep….and prove, with double arrow,…let's see if we are able to prove this theorem that if a is not equivalent to b then, double arrow, a minus b is not equivalent to zero.

347. *Tcl writes*

348. If a =/= b <=> a-b =/= 0

From the discussion of the case of "A not equivalent to B", which was originated by the introduction of the axiom theorem in l'Algebrista, a new theorem (out of the world of L'Algebrista) is originated and it is proved (we omit the proof), using also the axiom theorem. After this new theorem is proved, a new button is created, and it takes the status of "theorem button":

Excerpt 10 (class 2000)

481. T: now, as we have the theorem, then we have a new button that will not be an axiom button, but will be a theorem button […]

In the rest of the episode the teacher asks pupils to produce new theorems/buttons, in order to use them as instruments to solve equations.

Excerpt 11 (class 2000)

492. T: […] there must be some other theorem for equations, not only those of these two buttons, maybe you can find out others, I mean, it may be the case that if a is equivalent to b, then there is something more apart from the fact that the difference is zero, right? And these new buttons (*the ones pupils are being asked to invent*) maybe be useful to solve equations, so I am asking you, as a homework is […] to think of using some new buttons to solve equations.

Finally we observe that here the class has not yet reached a final decision concerning the status of the *axiom theorem*, However, it was decided to keep it undecided and to use it both as an instrument for solving equations, and as a means for deriving other principles to be used as intruments for solving equations.

9.4.5. Instrumental aspect of the meaning of *Theorem*

As already said, Teacher's intervention [237], aims at giving the axiom theorem the status of a button, this has an immediate implication: because it has become a button, it is now officially an instrument that pupils can use in their future activities; this is explicitly stressed by the teacher:

Excerpt 12 (classe 2000)

409. T: so we have our axiom theorem, no one can private us of it, and we even made a button for it, there it is (*points to a writing on the blackboard representing the button of the axiom theorem*), thus we can use it […]

in mathematics, A given theorem is not just a "valid statement", that is a statement of which a proof has been provided, but it is also an instrument that can be used to prove other theorems or solve problems. All that has a counterpart in the microworld, where, in the fact, a button can actually be physically used as an instrument to accomplish a certain goal. The teacher's intervention aims at exploiting such a parallelism between button and theorem in order to foster the operative meanings of the latter. the correspondence between buttons of the microworld and algebraic theorems (or axioms) constitutes the base for the semiotic game, thus it can be exploited as in the specific case of this episode.

Similarly to theorems, also axioms can be interpreted either as elements of a theory, or as instruments for accomplishing algebraic activities. In the episode discussed above ([409], such polysemy, was exploited by the teacher, when she asked pupils to produce a new button on the basis of the axiom theorem, thus enriching it with operative meanings. In the following we are going to discuss in what sense, and how, the teacher exploits the polisemy of some specific word in order to foster the theoretical and instrumental meanings associated to axioms or theorems. We will discuss the case of theorems, similar arguments can be brought for the case of axioms.

A word such as "theorem" is polysemic in the sense that it can be interpreted as referring both to an element of a theory (a statement and its proof), and to an instrument for producing new elements, i.e. proving new statements. The first meaning concerns the status of a given statement proven within a theory (theorem as an element of a theory), the other concerns the instrumental function of a given element of a theory when it is employed as a means for proving the validity of other elements (theorem as instrument). In order to foster both these meanings, associated to the word "theorem", we use L'Algebrista, and the *class algebra notebook*, as semiotic instruments allowing us to split the word "theorem" into two different words "theorem" and "button". Such a split is possible thanks to the contemporary presence of the world of Algebra (represented by of the class algebra notebook) and the world of L'Algebrista. At the beginning of the activities, the words "axiom" and "theorem" are referred to the elements of the class theory and related to the activity of editing and update of the class algebra notebook; whilst the word "button" is referred to the commands of the microworld, and associated to corresponding activities of transforming expressions in the microworld. In this way, for each of the two meanings of the word "theorem" considered, there are two dedicated word, and a dedicated class of activities. Thanks to L'Algebrista and the class algebra notebook, it is possible to foster both meanings separately, as they are rooted in two different practices and are represented by two different words. However, our educational aim is not to keep such meanings as separated, instead we want them to merge back into the word "theorem" at the and of the instructional sequence. For such a reason, a link is built *on purpose*, along the sequence of activities, between the "world" of Algebra (which includes the class class algebra notebook), and the (micro)world of L'Algebrista; this is done by passing countinuosly from one world to the other, so that the two meanings and the two words "theorem" and "button" can finally merge in the single word "theorem" with the two meanings.

This process is long and may take time to be accomplished. In the episode that we are analysing below, it is showed an intermediate step in which we assist to a movement from the world of Algebra to the world of L'Algebrista. The use of the hybrid expression "Theorem button" (ita.: "Bottone Teorema"), witnesses the dynamical process of merging of the two meanings, one referring to the element of a theory and the other to the instrument, highlighting the contribute of each of the two world to the fostering and mearing of such meanings.

As for the case of the polysemy of symbol "=", that was split into two component, by means of adding the symbol "= =", here the polysemy of the words "axiom" and "theorem" was split by means of adding the words "axiom button " and "theorem button". These hybrid words can be used either to bring instrumental meanings to the words "axiom" and "theorem" (derived from practice in L'Algebrista) either to bring theoretical meanings to the word "button", exporting in L'Algebrista a theoretical classification of its buttons. The words "axiom button" and " theorem button", result to be the junction nodes between buttons and axioms theorems. This semiotic game clearly appears in the following teacher's intervention [409] , drawn from the same discussion previously presented:

Excerpt 13 (class 2000)

481. T: because we have a theorem, we have a new button which is not going to be an axiom button, but it will be a theorem button […]

The theoretical meaning of the word "theorem" (as element of the class theory, and of the *class algebra notebook*), and the instrumental meaning of the word "button", are merged in the word "theorem button" that has the poslysemy that the teacher aims to foster for the word “theorem”. In fact the word “theorem button” represents at the same time a “theorem” and a “button”. It is a hybrid word that refers both to the world of Algebra and to the world of L’Algebrista, constituting a semiotic link between them, and as such it may function as a *pivot* for directing the focus of discurses from one world to the other. This is what happened in a excerpt that we are now going to analyse.

The episode takes place at the end of the collective discussion that we have been analyising in this section. The episode begins with the teacher recalling an operation performed by Cri in a previous geometry lesson, while proving a statement concerning the angles of a geometrical figure, she transformed **a+b+g = p** into **g = p-(a+b)** Such a transformation, is then interpeted in terms of summing **(-1)(a+b)** to the two terms of the equality, and then cancelling opposite elements (by means of the related theorem proven by the class). Once this transformation is explained verbally, the teacher proposes to transform it in a “button”, that is an instrument that can be used whenever it is necessary:

Excerpt 14 (class 2000)

512. T: […] so, how could we translate into a button this thing, Cri? […]

The request of generalisation of a practice happens through the request of creating a related button, that is of building an instrument to be used for such practice. However, the goal is also to introduce a new related theorem:

517. T: […] but what is this? What do you aspect this to be? Axiom button? Theorem button?

519. Gst: theorem button!

519. T: theorem button, thus we are going to prove it

At the direct request of the teacher to give a status to the new button (517), the new button is referred to as "theorem button", using this name the pupils recognize that the new button must have a correspondent in a theorem of the theory, thus “it needs to be proved”, like any theorem. The word "theorem button", thanks to its polysemy, functions as a pivot, allowing movements back an forth between L’Algebrista microworld and the Algebra theory, the sphere of practice and the sphere of theory. Thanks to the polysemy of such a pivot word, theoretical meanings are conveyed to the sphere of practice, and vice versa, .

9.4.6. The interplay between two worlds

The episode we showed presents a cycle of production/evolution of meanings that was originated by a voluntary (directed by the teacher) interplay between two worlds: the world of L'Algebrista, and the world of algebra, outside L'Algebrista[49]. Drawing from the above discussion in this section we are going to sum up what movements the teacher directed from one world to the other and vice versa.

[49] Here we don't intend to give a strict definition of world, this is just a way to differentiates between class practices, and class knowledge, that are related directly to L'Algebrista and those that are not.

Figure 23 presents a scheme of what happened in the episode. During the first part of the class discussion, the axiom theorem was recalled, and was related to the solution of equations. During this first part there is no evidence of any reference to L'Algebrista, thus we placed it on the side of *algebra world*. Teacher's intervention [237] explicitly shifts the focus on *L'Algebrista world* and results in production of a hypotetical[50] new **button,** corresponding to the **axiom theorem**. One of the consequences of such shift is the problem raised by Cri [273-289], she is wondering what happens if A and B are not equivalent, she is wondering how L'Algebrista could be able to do it; the functioning of button, due to the nature of the software, cannot correspond perfectly with the functioning of theorem originated in mathematics.. At this point the teacher takes this problem, originated within *L'Algebrista world*, and brings it back into *algebra world* [340,342], originating a brief discussion that leads the class to state and prove a new theorem corresponding to the transformation rule for equation expressed by the formula “A=/=B<=>A-B=/=0”. The last theorem is finally brought into L'Algebrista in the form of a new button [481]. From an algebraic point of view, this new theorem is not strictly needed, because it is an immediate consequence of the theorem represented by the “axiom theorem”. Nevertheless, because it was originated within the class discussion, the class decided to give it the status of a theorem of their theory, and decided create a corresponding button. And this is the interesting point for our analysis.

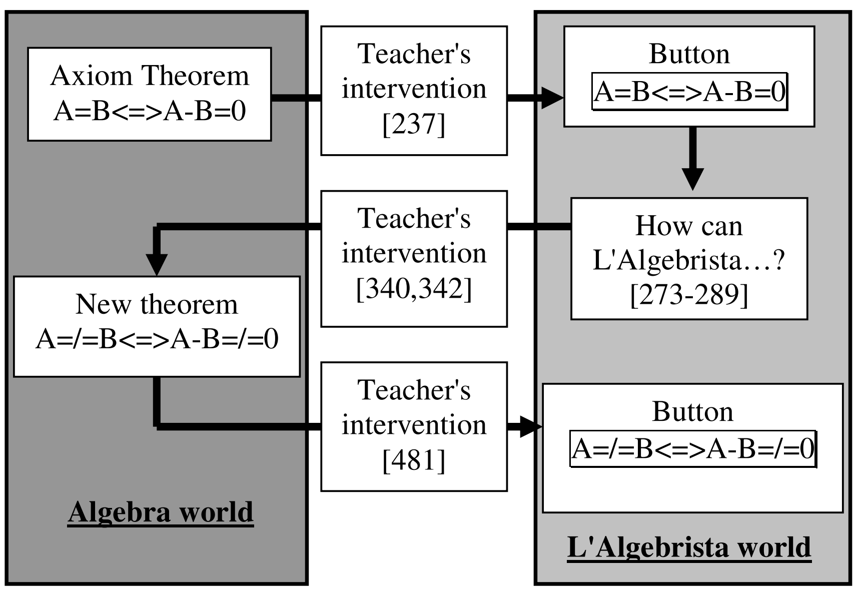


Figure 23 The interplay between the two worlds

9.4.7. linking the two worlds

In summary, The interventions of the teacher consist mainly in forcing movements back and forth between the two worlds. However, during the discussion, it is possibile to identify another important kind of intervention. Consider the following intervention of the teacher [481]

[50] We recall that at the moment of the analysed collective dicsussions computers are not available

Excerpt 13 (class 2000)

481. T: because we have a theorem, we have a new button which is not going to be an axiom button, but it will be a theorem button […]

Here the teacher explicitly recalls the correspondence between axioms (and theorems) and buttons. This intervention is situated neither in the world of L'Algebrista nor in the Algebra world, rather it is situated at a metalevel where the relationship between the two contexts has to be elaborated. This intervention refers to what we can call the "linking knowledge", concerning the relationship between the two worlds. Such relationship is not a perfect correspondence, and to "know" means to control the shift of meanings to be accomplished when moving between the two worlds. In the case of intervention [409] (Excerpt 12) the correspondence is exploited in order to bring operative meanings from L'Algebrista to the world of algebra. On the contrary, in Excerpt 9, we observed that Cri's uneasiness (excerpt 9, [273-289]) arises because the pupils realises a discrepancy between the two worlds. During the discussion the software is not available, and Cri's reasoning is based only on her knowledge about what she can do inside L'Algebrista.

In conclusion when the teacher moves from one world to the other, she may exploit not just each single world separately, but also the relationship between them, in order to guide the development of mathematical meanings, as separated from, but related to, meanings concerning the microworld.

The analys of the discussion has shown how the teacher used different elements of L'Algebrista, as instruments of semiotic mediation, exploiting "communication strategies aimed at guiding the evolution of meanings within the class community" (**[51]**, Mariotti, 2002). The interventions we individuated are based on the distinction between two worlds, that of L'Algebrista and that of class mathematics (Algebra), and on the relationships between them. The relationship, and distinction, between the two worlds, is managed by the teacher also by using hybrid signs (for instance "theorem button" and its meanings) characterised by a polysemy that make it possible –both for the teacher and the pupils - to talk about both the two worlds and to compare them. Movements from *L'Algebrista* to *algebra theory* (and vice versa) are possible only if the participants to the activity are conscious that there is a distinction between them, and Cri's example shows how such movements can be effective when pupils are familiar with such distinction.

As a consequence we hypothesise that one key point of an effective use of an artefact as instrument of semiotic mediation is to set up activities with focus on the relationship between the *world of the artefact* and the *world of the mathematics we want to teach*. Moreover it seems relavant to identify particular expressions, whose polysemy allows them to function as "pivot word", fostering the movement between the two worlds. In the previous examples, the relationship between the two worlds was eploited by the teacher by means of the polysemy of words like: one derived from the sign of the microworld, i.e. "insert expression", the other coined expressly by the class, "theorem button". these words were used by the teacher (but sometimes also by the pupils) to direct the focus of the discourse and to direct the nature of practices that had to be brought forward.

The teacher plays a key role in individuating and stressing differences and analogies / similarities between the two worlds: both differences and similarites can be exploited in order to guide the construction and evolution of mathematical meanings. The example of the functioning of the pivot word "axiom theorem" can be considered paradigmatic.

10. Conclusions

10.1. Microworlds and artefacts as instruments of semiotic mediation

Research on the use on technological devices for educational purposes, has been showing potentialities and limits. One of the most interesting ideas that we find in literature is that of Microworlds, environments where it is possible to experience activities that are relevant to a an incorporated knowledge domain. Thanks to microworlds, learners are given the possibility to experience, phenomenologically, knowledge domains such that of mathematics, which otherwise is perceived as abstract, and far from pupils practical experiences. Thanks to microworlds, it is possible to set up mathematical practical fields of experience, that can be exploited for educational purposes. Activities in a microworld may result in learning outcomes that are relevant to the considered knowledge domain. However, research showed that, even if practices with mathematical microworlds result always in some learning outcomes, the knowledge learnt by the pupils may not always coincide with the teacher's intentional mathematical knowledge. As a matter of fact, a learner working in a microworld, learns knowledge concerning it, but the relationship between such knowledge and the teacher's intentional knowledge is not straightforward. In fact, if no specific interventions are set up, it is not even guaranteed that the pupil interprets the activity with the microworld as a mathematical one.

In the case of microworlds, as for in the case of other artefacts used in educational research, it is not obvious that pupils interpret activities as mathematical ones, and in case they do, it is not obvious that their learning outcomes are consistent with mathematics. Research showed that the nature of specific microworlds, or other artefacts, themselves, is not enough to guarantee the consistency of learning outcomes with a teacher's intentional mathematical knowledge. This is essentially because the way a user perceives an artefact or a microworld, and activities with them, is not always foreseeable and may be not consistent with a teacher's plans.

The problem that we described has been addressed by some researchers within the framework of semiotic mediation ([51], Mariotti, 2002) who present approaches in which learning outcomes are considered to be rooted in the phenomenological experience, but that can reach consistency with mathematics, thanks to an evolution guided by the teacher by means of peculiar communication strategies. Within this framework a key role is played by the idea of instrument of semiotic mediation, which refers to a special use of instruments in class practices: the instrument is introduced in the practices on purpose by the teachers, and it is exploited to accomplish communication strategies that aim at developing meanings related to the mathematical contents consistent with the motive of the teaching/learning activity. Within this framework a key idea is that of deriving, from a used instrument, hybrid signs which refer both to the sphere of practice with the instrument, and to the sphere of theory of the teacher's intentional mathematical knowledge. Such hybrid signs can be used as pivots for directing the focus of activities and discourses either toward the sphere of practice or toward the sphere of theory. Movements back and forth the two spheres are exploited to convey practical meanings to the sphere of theory and theoretical meanings to the sphere of practices, allowing to exploit microworlds as means for generating theoretical meanings. In this framework, how theoretical meanings are originated from phenomenological experience depends strictly on how the teacher exploits hybrid signs as pivot, structuring a complex relationship between the considered microworld or artefact, and the mathematical intentional knowledge. Meanings are developed under the guidance, thus under the control, of the teacher who is institutionally in charge of ensuring their consistency with mathematical knowledge.

Within this framework, one of the objectives of this thesis was to study and to improve the formulation of teaching/learning practices that exploit microworlds as instruments of semiotic mediation.

10.2. A theoretical approach to algebra

The knowledge domain chosen for our research was that of algebra, and our educational goal was that of introducing ninth grade pupils to symbolic manipulation. Research on the subject showed problems with standard educational approaches for what concerns this knowledge domain. Particular highlighted problems are that of difficulties for pupils to move from the operational perspective, typical of arithmetic, to the structural perspective typical of algebra. A key point in such a passage is that of considering algebraic expressions not only as computation procedure to be executed, thus processes, but also as objects to be acted upon; the algebraic way to act upon expression is that of manipulating, and transforming them by means of a set of axioms, definitions and theorems. Such kinds of algebraic manipulations are based on the notion of equivalence of expressions, which can be defined either in terms of numerical computations, or in terms of axioms based transformations. As consequence it is possible to set up an approach to symbolic manipulation interpreted as an activity of comparing expressions and proving their equivalence, or non equivalence, by means of algebraic axioms, definitions and theorems. Within such an approach, expressions are not only computational procedures, but are also objects that are compared and consequently manipulated according to goal oriented transformations; the goal of such activities is thus to state on the equivalence of these objects, differently for what happens in arithmetic, where the goal of computations of expressions is to get a final numerical result to be used to solve other external problems. In our research we chose to follow such a theoretical approach to algebra, which consequently leads us to define the second main educational goal of our study, that of introducing pupils to theoretical thinking.

Within to the chosen vygotskian theoretical framework, we set up an educational approach based on the use of microworlds in order to introduce pupils to theoretical thinking and to symbolic manipulation.

10.3. A software specially designed for introducing pupils to algebra as a theory, within the framework of semiotic mediation

Many researchers have been conducted on the domain of computers and algebra education, however, none of them is based on interpreting symbolic manipulation within a theoretical perspective. At the same time, no educational software had been implemented to introduce pupils to algebra and to theoretical thinking at the same time. As a consequence, for our research, we chose a theoretical framework based on the notion of semiotic mediation, within which we find examples of experiments focused on introducing pupils to theoretical thinking, in geometry, using microworlds such as Cabri. According to the hypothesis of this theoretical framework, the meanings rooted in the phenomenological experience within the microworld can evolve toward a teacher's intentional mathematical knowledge under his/her guidance, by means of specific communication strategies. Within this framework, we realized a specially designed software, L'Algebrista, where it is possible to transform expressions by means of commands (buttons) that correspond to axioms, definitions and theorems of algebra (see chapter 4.). The practice within this microworld can be interpreted in terms of activities of proving equivalencies of expressions by means of the elements of a theory represented by axioms corresponding to the set of available buttons. Such a practice constitutes a phenomenological experience originating meanings that can evolve toward mathematical meanings related to, and consistent with, the teacher's intentional algebraic knowledge. The elements constituting the software had been designed so that their relationship with corresponding elements of algebra theory is the most direct possible, in the sense that we tried to avoid commands that incorporate many axioms or theorems at the same time. Thus the relation between the commands of the software and the considered axioms, definitions and theorems, is almost biunivocal. This has been done because we wanted it to be possible to interpret L'Algebrista's set of commands, as signs standing for the elements of an algebraic theory of expressions. It is then possible to exploit its relationship with mathematics, as a means for building mathematical meanings rooted in the practice with L'Algebrista. If this is the objective, then it is important to have a clear picture of the relationship of the microworld with algebra, of the knowledge incorporated in the software, and of the knowledge we foresee it can evoke. In chapter 5. we presented a detailed analysis of the knowledge that was incorporated in L'Algebrista by its creators, and we presented hypothesis of what is the algebraic knowledge that can be evoked by the software and that is consistent with our educational aims.

10.4. The experimentation

Since 1998 we set up several experiments involving L'Algebrista, which evolved in itinere, together with our research, according to the methodology of research for innovation. The first year of experimentation is to be intended as a study of feasibility, which resulted in a better formulation either of the sequence of activities proposed to pupils, either of the educational strategies to be employed by the teacher to guide teaching/learning processes.

In chapters 6. and 7. we give a description of the experiments firstly in terms of the general teaching/learning paradigm and then in terms of the key steps of the sequence of proposed activities. Both the paradigm and the sequence of activities, evolved in itinere along with the experimentation, and are to be intended as a result of this research.

10.4.1. The key steps of the sequence of activities

The sequence of activities we presented to pupils is characterized by the following key steps:

- *Introduction of the idea of proving equivalence relationships of numerical expressions by means of the axioms of a theory*: instead of generalising arithmetic to literal expressions, we introduce a new practice for numerical expressions, an algebraic one, centred on goal oriented transformations by means of the elements of a theory.
- *Extension to literal expressions of the practices of equivalence proving by means of axioms*: once such algebraic practices are established on numerical expressions, they are extended to the case of literal expressions;
- *Proving new theorems*: due to the generality of literal expressions (each letter can represent an expression), it is possible to interpret the proof of the equivalence of two given expressions, as the proof of theorem stating that any two expressions having the same structures of the given ones, are equivalent. As a consequence, among the many proven equivalencies, some are chosen to be given the status of theorem of the theory of reference that the class is building. The new theorems can be used as means for proving other equivalencies; in fact we introduce many common transformation rules as theorems, for instance rules for summing or multiplying fractions, rules for managing powers, and rules for facilitating factorisation of expressions.
- *Equations*: when comparing two expressions, we have two main cases: the two expressions are equivalent or not equivalent; in the first case we can prove it by means of axioms (or theorems) based transformation. In the second case, we prove it by substituting numbers and computing the obtained numerical expressions; if the expressions are non equivalence, then there are some numbers that substituted to the letters originate non equivalent numerical expressions. However practice shows that sometimes non equivalent expressions, for certain numbers substituted to the letters, originate equivalent numerical expressions: the problem of equations consists on individuating such numbers. Also the principles for solving equations can then be introduced within a theoretical perspective, for instance taking the rule "A=B <=> A-B=0" as an axioms, it is possible to easily prove the rule "A=B <=> A+c=B+c"; this is exactly what we do in our experiment.

10.4.2. The general teaching/learning paradigm

In our experimentation all the activities are accomplished either in the paper and pencil environment, or in L'Algebrista. In fact the microworld has been designed on one hand for performing axioms or theorems based transformations, on the other hand for being enriched with new commands, corresponding to new theorems, and added to the software by the users. In other words not only it is possible to use buttons representing axioms as instruments for transforming expressions and proving theorems, but is also possible to create new buttons, corresponding to theorems, and that can be themselves used for transforming expressions and proving new theorems.

As consequence it is possible to develop the key steps of our sequence of activities either working in paper and pencil, either in the microworld. One of the aims of our research was to study how such parallelism could be exploited, and how, starting from the practices with L'Algebrista, it is possible for the teacher to guide pupils' learning toward her algebraic intentional knowledge. We started from the basis posed by previous research on the use of microworlds as instruments of semiotic mediation in the case of geometry, from which we pursued the key ideas of a teaching/learning paradigm that had proven to be effective. Consequently, the paradigm (described in detail in chapter 6.) we set up for our experiment, consists of a cycle of activities developed either with L'Algebrista, or with paper and pencil, interwoven with class discussions.

According to our hypotheses, the meanings, raising from phenomenological experiences within the microworlds, have to evolve, under the guidance of the teacher, towards the mathematical meanings the teaching/learning activity aims to. In our teaching experiments, the main structure of class activities can be schematised as shown in Figure 24.

Meanings originated in the phenomenological experience are shared within a collective discussion, fixed in the sets of command of Cabri and L'Algebrista and then reported in the personal *notebook*. Practical activities are verbalised in the forms of written reports and class discussions leading to the production of the class notebook and update of the commands of the microworld. The notebook and the sets of command of the microworld, are then cyclically revised in order to formulate their logical structure in terms of the logical relationship between the axioms and the theorems of a theory.

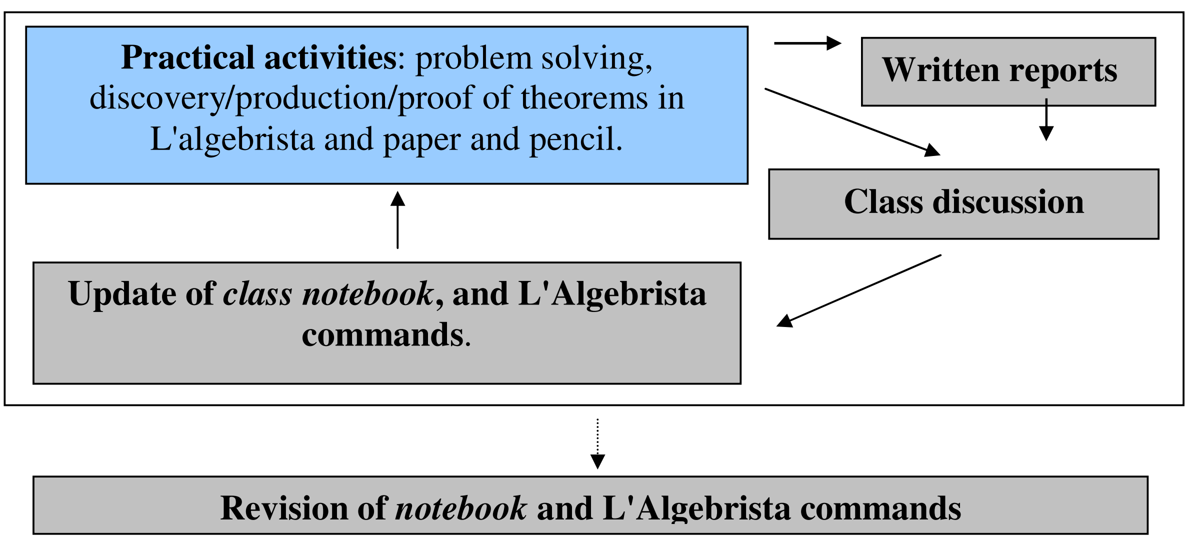


Figure 24 The main structure of the class activities: practical activities are verbalised in the forms of written reports and class discussions leading to the production of the class nothebook and update of the commands of the microworld. The the notebook and sets of command of the microworld are then cyclically revised in order to formulate their logical structure in terms of the logical relationship between the axioms and the theorems of a theory.

The main elements constituting the paradigm are: L'Algebrista, the paper and pencil environment, class discussions, and the mathematical notebook. The roles played bye such elements, are defined in terms of the functioning of L'Algebrista as an instrument of semiotic mediation, which is strictly tied to the idea of exploiting the polysemy of the Italian word "calcolo" (en.: "computation activities" or "symbolic manipulation"). Such a word (*Calcolo*), among other meanings, is used to refer to algebraic computations, including both, numerical computations, and rule based computations. *Calcolo*, in general, refers to both, arithmetical and algebraic ways for handling expressions. However, such a word is associated mainly to the activity of "calcolare espressioni" which in the case of numerical expressions means "to compute the numerical result" whilst in the case of literal expressions means "to expand and simplify the expression". In both cases one can proceed either computing the results of operations between numbers (when possible), or by using computational rules derived from the properties of the operations. In standard Italian approaches, pupils experience a lot of *Calcolo* with numerical expressions, then they are presented literal expressions, and asked to "calcolare" them, that is, the *calcolo* of numerical expressions is extended to literal expressions. This can result in pupils' difficulties when the differences between the numerical case and the literal case are not enough highlighted, which is often the case when algebra is presented as generalized arithmetic.

The key idea of our approach is situated exactly at this point, in fact, if we want to introduce symbolic manipulation within a theoretical perspective, the main focus of our activities has to be on transformations of expressions by means of the axioms of a theory. In other words we propose to interpret the rules based transformations of the *calcolo* as transformations based on the properties of the operations, which we take as the axioms of our algebraic theory. If in previous pupils experience the most relevant meanings of the word *calcolo* were the arithmetical ones, with the introduction of literal expressions, we wanted to stress the algebraic meanings of *calcolo*, fostering their evolution toward a theoretical view of symbolic manipulation. As a consequence we wanted to distinguish clearly the two different meanings associated to the word *calcolo* in order to avoid confusion and foster the evolution of both meanings within a theoretical perspective.

The first step of our intervention was thus to introduce, through a class mathematical discussion, the idea of comparing numerical expressions questioning their equivalence relationships. Numerical expressions could then be compared either by means of numerical computations either by means of transformations based on the properties of the operations, which we, on purpose, began to call also "axioms". This kind activity is the core of the activities proposed to pupils, and substitutes the activity of "calcolare" ("compute numerical results" or "simplify") with which pupils were familiar prior to begin the experimentation. The word *calcolo* and the activities of *calcolare* are on purpose eliminated, at the beginning of the experiment, from class practices. As we said, we wanted to separate algebraic handling of expressions from arithmetical handling of expressions, as a consequence, given the mathematical problem of comparing expressions, we introduced two new words: *verificare* and *dimostrare*, which can be translated with "to check" and "to prove". The meanings of the new words, as they had been introduced in class practices, are strictly tied to the idea of comparing expressions in terms of their equivalence relationship. In fact two expressions, in our experiment, are considered to be equivalent if either their numerical results are the same, or if it is possible to transform one into the other using the axioms of the chosen theory. Once these definitions are discussed and shared by the class, the teacher introduces the new words *to check* and *to prove* (ita.: "*verificare*" and "*dimostrare*") as referring to the two ways to define equivalencies of expressions. Thus, *to check* that to expressions are equivalent means to compute their numerical results and to check if the obtained numbers are the same, whilst *to prove* that two expressions are equivalent, means to transform one into the other by means of the given set of axioms[51]. These new words, introduced on purpose, structure a distinction between the arithmetical and the algebraic

51 Of course, the chosen axioms are discussed in class, we generally begin considering only the properties of the operations, and then add gradually other needed axioms to the theory, as we explained in chapter 8.

meanings of the word *calcolo* which is now split respectively into *check* and *prove* (or *proof*). This distinction is forced, on purpose by the teacher, and it is to be considered as temporary. In fact, the meanings fostered through this distinction and following activities, are meant to be merged again in the word *calcolo* once the experimentation is over. Our objective is that of creating solid algebraic meanings, as opposed to arithmetical meanings, and that pupils internalised them as such; after that such meanings can merge in the polysemic word *calcolo* which pupils should be able to manage with its different meanings. The process that leads to an integrated interpretation of the word “calcolo” as referring both to arithmetical computations and algebraic manipulations, is supposed to be a long term one, and we couldn’t study, along this research, its complete evolution, thus further research is needed.

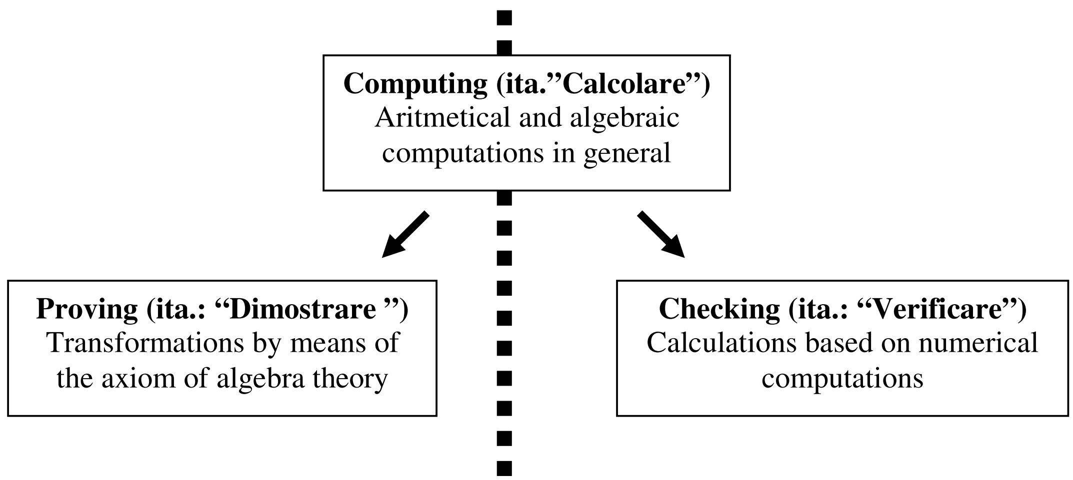


Figure 25 The semiotic split of the sign /computing/ into the signs /checking/ and /proving/

Once the sign /calcolo/ is split into the two signs /verifica/ and /dimostrazione/ (/check/ and /proof/), the L’Algebrista, is introduced in the class practices. In fact, the microworld, due to its features, can play a key role in keeping the separation between the meanings associated to the words “check” and “prove”. The activities of “proving” and the activities of “checking” are performed, in the microworld, by using distinct commands, the “buttons of the properties of the operations” in the case of proving activities, and “buttons of numerical computations” in the case of checking activities. As a consequence, L’Algebrista can be a source for phenomenological experience rooting meanings related to both the activity of proving and the activity of checking. However, as we previously observed, what a pupil may learn by using a microworld, is not necessary consistent with the teacher’s intentional knowledge. Thus there is a need for the teacher to have some means for controlling, and guiding, the evolution of meaning originated in the L’Algebrista toward meanings that are consistent to her/his intentional knowledge, and that at the end should not depend strictly on the microworld. In our approach, L’Algebrista is not supposed to be identified with the algebra theory corresponding to our educational goals, thus we need to create, in class practice, a separate context that can be identified with such algebra theory. In order to do that we introduce another semiotic split: the word “prove”, is split into “prove within L’Algebrista” and “prove”, while the word “check”, is split into “check within L’Algebrista” and “check”. The main environment for “checking” and “proving” is then the paper and pencil and refers to mathematics.

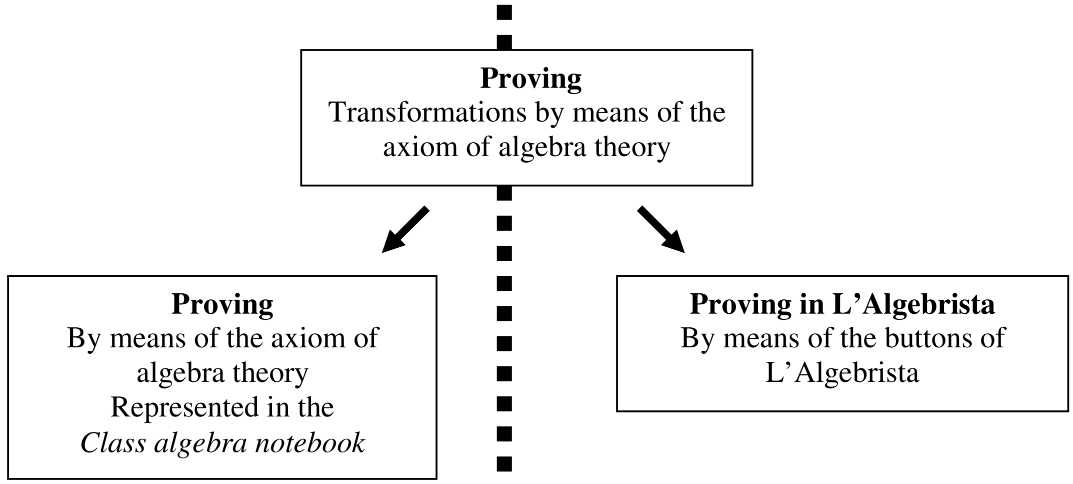


Figure 26 The semiotic split of the sign /proving/ into the signs /proving/ and /proving in L'Algebrista/

Now, in the case of "checking", pupils already had mathematically consistent meanings associated to it, thus the interesting case for us is that of "proving".

The idea of "proving" that we are trying to foster within our experiment is that of "proving within a theory", thus, every step of a proof has to be produced by means of the elements of the theory in question. In *L'Algebrista* the "elements of the theory" that can be used as means for proving are represented by the available buttons. As a consequence, if we want to develop meanings related to proving outside the microworld, we need to foster the idea of theory as a set of axioms (and then theorems). In order to foster this idea, together with L'Algebrista, we introduce the *class algebra notebook*, which is meant to be a collection of all the axioms and theorems shared by the class, and each pupil is supposed to keep his/her own copy. In other words the *class algebra notebook* represents the algebra theory shared by the class. At this point, *proving* outside L'Algebrista makes sense in terms of proving by means of the axioms and theorems contained in the *class algebra notebook.*

L'Algebrista and the *class algebra notebook* are presented as related but not as the same thing. Key elements of the idea of theory, such as axioms and theorems with their proofs, are represented both within L'Algebrista and in the *class algebra notebook*. The relationship between the *class algebra notebook* and L'Algebrista is exploited in order to originate meanings in L'Algebrista, guiding their evolution toward algebraic meanings represented in the notebook. In section 9.4.5, for instance, we showed how this relationship can be exploited by the teacher to foster two peculiar meanings associated to the word "theorem". A theorem, in fact, is both an element of a theory, and an instrument that can be used to prove other theorems. In order to foster both these meanings, we exploit the semiotic split between "proving" and "proving in L'Algebrista". In fact we split the word "theorem" into "theorem as element of the *class algebra notebook*", and "button in L'Algebrista". In the first case, the associated meaning is that of element, proven to be valid, of the class theory; in the second case the associated meaning is that of instrument that can be used to prove equivalencies in L'Algebrista.

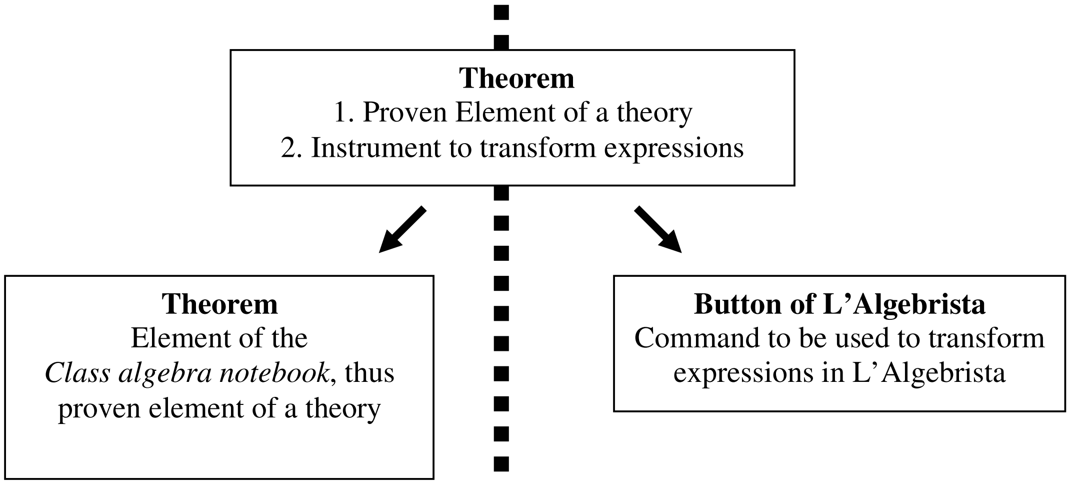


Figure 27 The semiotic split of the sign /theorem/ into the signs /theorem/ (of the class algebra notebook) and /button of L'Algebrista/.

The two meanings associated to the idea of theorem can thus be fostered on the one hand by practices of edition and revision of the *class algebra notebook*, and on the other hand by practices of proving theorems in L'Algebrista. However, our aim is that pupils interpret the word "theorem" both as an element of a theory and as an instrument for proving, thus we need to merge together the two meanings associated to theorems of the class algebra notebook and to the buttons in L'Algebrista. In section 9.4.5 we presented an episode in which the teacher guides the merging of these meanings by deliberately using the hybrid word "Theorem button", which is used as a pivot to direct the focus of the discourse from the world of L'Algebrista to the world of algebra and vice versa. This a hybrid word conveys both the mentioned meanings associated to the idea of theory, but keeps track of how these meanings, in class practice, have been originated in the two different contexts. We hypothesise that hybrid words like this one can be the key for guiding the evolution of meanings that are originated in the microworld, toward mathematical meanings represented, in our case, by the *class algebra notebook*. The word "theorem button", in the final phases of the experiment, is gradually substituted by the word "theorem", as the microworld is gradually abandoned. In fact, as pupils internalise the use of the "theorem buttons", they begin to feel the need to abandon L'Algebrista, which in the final phases proves to be obsolete for them because of its limitations. As a consequence the microworld is abandoned, and what remains is the class algebra theory with its axioms, definitions, and theorems, as represented by the *class algebra notebook*.

The scheme in Figure 28 represents the evolution of the word "theorem" and the associated meanings. The word was deliberately split by the teacher into "theorem" (in the sense of "valid, and proven, element of the class algebra notebook") and "button"; we may call this phase *semiotic split*. The initial meanings associated to these two words evolved under the guidance of the teacher in two distinct ways: on the one hand, through activities of proving theorems both in L'Algebrista and with paper and pencil (using the elements of the class algebra notebook); on the other hand through activities of edition (insertion of new theorems) and revision of the notebook, and insertion of new buttons in L'Algebrista (with "Il Teorematore"). We may call this phase *meanings evolution*. Finally the meanings associated to the two words "theorem" and "button" are merged together by means of the hybrid word "Theorem Button"; we may call this phase *semiotic merging*.

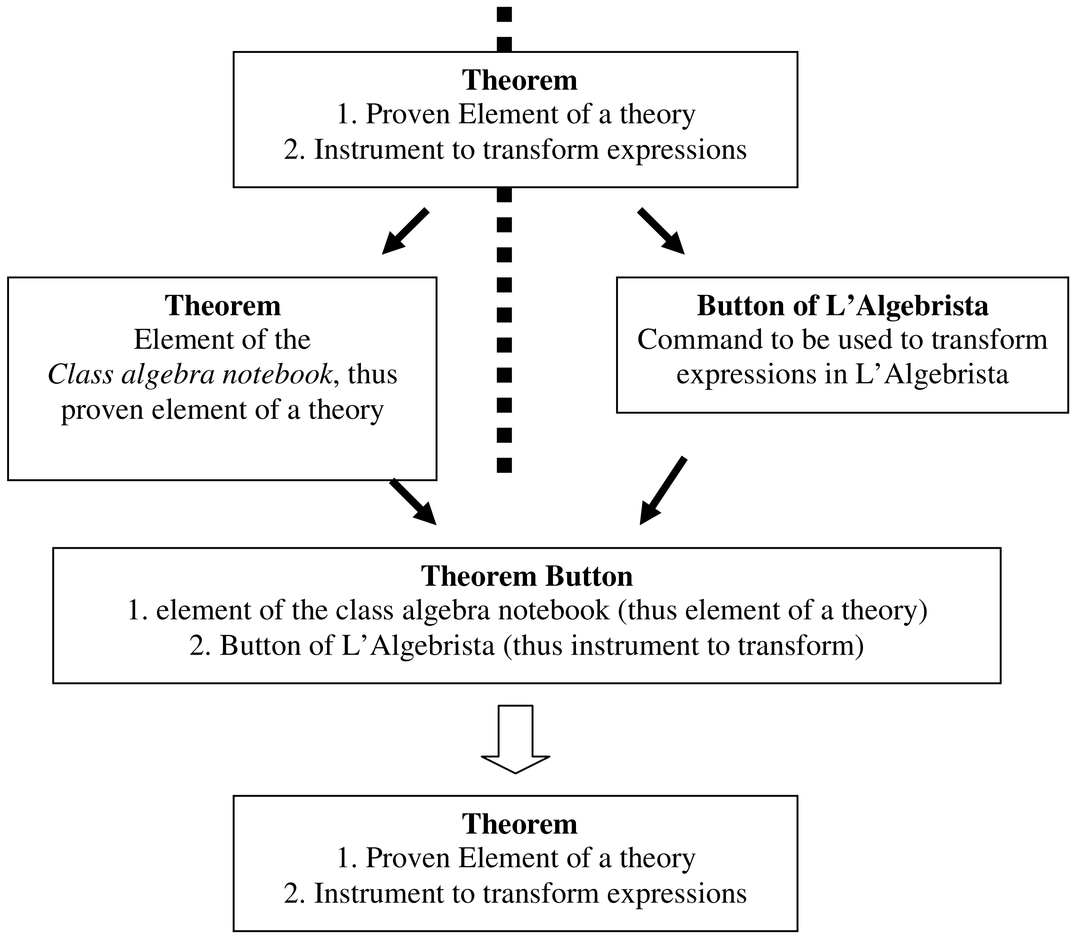


Figure 28 Thanks to the use of the hybrid word "Theorem button", the theoretical meanings associated to the theorems of the class algebra notebook, and the instrumental meanings associated to the word "button" are merged together, posing the bases for fostering the idea of theorem both as an instrument and as an element of a theory.

A Mathematical object can be interpreted both as an element of a systematized theory, and as a means for accomplishing tasks and solving problems, in other words it is associated to both theoretical meanings and to instrumental meanings. In our approach these two meanings are introduced separately starting from the introduction of the *class algebra notebook* and the microworld L'Algebrista. Different words are used to refer to elements of the *class algebra notebook* and their counterparts in L'Algebrista (for instance "theorem" and "button"), in this sense we operated a *semiotic split*. The meanings associated to the two words develop separately because they originate in different contexts, that of algebra, represented by the *class algebra notebook*, and that of L'Algebrista. However, the features of the microworld allow the teacher to guide the evolution of the two meanings, keeping them tied to each other, for instance by using hybrid words, such as "theorem button", that function as nodes between the world of algebra and the microworld. Thanks to such hybrid words it is then possible to close the semiotic split, merging the originated signs into a unique sign where a mathematical word is used to represent both the theoretical and the instrumental meanings that have been developed. At the end of the teaching/learning process, when the microworld is abandoned, hybrid words are gradually merged with, and substituted by, words belonging to the domain of algebra as it is represented by the class algebra notebook. The pupils of class 2000 in our experiment named this final phase the "death of L'Algebrista"; in fact they decided to abandon the microworld because they realised that in L'Algebrista it wasn't possible to prove certain statements that they were able to prove with paper and pencil referring to the elements of the *class algebra notebook*. Thus the world of L'Algebrista was finally encompassed in, and replaced by, the world of algebra.

We argue that a similar educational approach can be extended to other situations where the educational goal consists in introducing pupils to both the theoretical and the instrumental aspects of a given area of mathematics. A *class mathematical notebook* could be used to represent the mathematical theory shared by the class, and a microworld could be exploited to foster instrumental meanings to be associated to the elements of such a theory. However, further research is needed in order to identify what areas of mathematics this approach is suitable for, and in order to identify what kinds of microworld (or maybe other kinds of artefacts) can be used as instruments of semiotic mediation.

11. Appendix: The meanings of some key words

Here we report the definitions of some key words as they are reported in some dictionaries. Each section will be a list of definition for a word.

11.1. Objects and things

"**Object:** a material thing; that which is thought of or regarded as being outside, different from, or independent of, the mind (as opposed to subject); that upon which attention, interest, or some emotion is fixed; a thing observed; […]"

([23], The Chambers Dictionary, 1998)

"**Object:** *n.* physical thing; focus of thoughts or action; aim or purpose; […] article, boy, fact, item, reality, thing […] aim, butt, focus, target, victim […] design, end, goal, idea, intent, motive, objective, point, purpose, reason."

([24], Collins, pp. 418, 1994)

"**Objeto:** m. Todo lo que puede ser materia de conocimiento intelectual o sensible […]. ‖ Propósito, intención […]. ‖ Asunto, motivo […]."

([36], Garcia-Pelayo y Gross, pp. 393, 1983)

"**Oggètto:** s. m. **1.** Entità fisica o spirituale in quanto contenuto di un'esperienza o di un'attività (*l'o. della conoscenza, delle percezioni; l'o. dei miei studi*) che può identificarsi in un 'fine' (*l'o. di una ricerca, di un desiderio*) o in un 'argomento' (*l'o. del discorso)* | […] **concr.** Unità materiale distinta da una propria sussistenza per lo più di ordine quantitativo o qualitativo: *un o. pesante; o. artistici; aveva le tasche piene di o. inutili* | […] **2.** In ottica: *punto-o*. […] il punto da cui vengono (*o. reale)* o sembrano provenire (*o. virtuale)* i raggi che concorrono a formare l'immagine fornita da un sistema ottico. [dal lat. Mediev. *Obiectum*, neutron sostantivato di *obiectus*, participio pass. Di *obicĕre* 'metter di fronte'."

([27], Devoto et al., pp. 1531, 1971)

"**Thing:** *n.* material object; object, fact, or idea considered as a separate entity; […] affair, article, body, concept, entity, fact, matter, object, part, portion, substance […]. Apparatus, contrivance, device, gadget, implement, instrument, machine, means, mechanism, tools […]"

([24], Collins, pp. 621, 1994)

"**Cosa:** f. Palabra indeterminada cuyo significado (materia, objectos, bienes, palabras, acontecimiento, asuntos) se precisa por lo que la precede o la sigue […]. ‖ Ser Realidad, por oposicióon a aparencia […]. ‖ Lo que se piensa, lo que se hace, lo que pasa […]."

([36], Garcia-Pelayo y Gross, pp. 393, 1983)

"**Cosa:** s. f. **1.** Nome estremamente generico, che riceve determinazione solo dal contesto del discorso; oggetto ideale o materiale: *c. corporee, incorporee, temporali, eterne;* […] **part.** In filosofia (fino a Kant): *c. in sé*, ciò che sussiste indipendentemente dal nostro conoscere."

([27], Devoto et al., pp. 604, 1971)

11.2. Artefact

"**Artefact:** a thing made by human workmanship; […]"

([23], The Chambers Dictionary, 1998)

"**Artefact, artefact:** *n*. something made by man"

([24], Collins, pp. 418, 1994)

"**Artefatto:** agg. Adulterato, insincero. [dal lat. *Arte cactus* 'fatto con arte' e cioè 'con artificio']."

([27], Devoto et al., pp. 174, 1971)

11.3. Instrument

"**Instrument:** a tool or utensil; a contrivance for producing musical sounds; a document constituting a contract; a formal record; a person or thing used as a means of agency; a term generally employed to denote and indicating devices but also other pieces of small elecrical apparatus"

([23], The Chambers Dictionary, 1998)

"**Instrument:** *n*. tool used for particular work; object played to produce a musical sound; measuring device to show height, speed, etc.; *Informal* person used by another. […] appliance, contrivance, device, gadget, implement, mechanism, tool, utensil"

([24], Collins, pp. 324, 1994)

"**Instrument:** […] a means whereby something is achieved, performed, or furthered […]"

([58], Merriam-Webster, 2003)

"**Instrument:** n. 1. A means by which something is done; an agency. 2. One used by another to accomplish a purpose; a dupe. 3. An implement used to facilitate work. See Synonyms at tool."

([78], The American Heritage Dictionary of English Language, 2000)

"**Instrument:** 1. That by means of which any work is performed, or result is effected; a tool; a utensil; an implement; as, the instruments of a mechanic; astronomical instruments."

([84], Webster's Revised Unabridged Dictionary, 1996)

"**Instrumento:** m. Aparato, utensilio o herramienta para realizar trabajo || Aparato para producir sonidos musicales […]. || Escritura con que se justifia una cosa […] || Fig. Lo que se emplea para alcanzar un resultado […]."

([36], Garcia-Pelayo y Gross, pp. 304, 1983)

"**Strumento:** [..] s. m. **1.** Arnese indispensabile per lo svolgimento di un'attività, di un'arte, di un mestiere […]. **3.** Mezzo di cui ci si può attivamente servire per il conseguimento di uno scopo: *della penna si fece s. di lotta; non vorrei essere s. dell'ambizione altrui* […] ."

([27], Devoto et al., pp. 2386-2387, 1971)

11.4. Tool

"Tool: a working instrument, esp one used by hand; the cutting part of a machine tool; someone who is used as the mere instrument of another; anything necessary to the pursuit of a particular activity; any of several devices used to impress a design on a book cover; an impressed design on a book cover; a weapon, esp a gun; the penis; a utility, feature of function available as part of a word-processing packgage or database"

([23], The Chambers Dictionary, 1998)

"Tool: *n.* implement used by hand; person used by another to perform unpleasant or dishonourable tasks. […] appliance, contrivance, device, gadget, implement, instrument, machine, utensil […]"

([24], Collins, pp. 627, 1994)

"Tool: Something regarded as necessary to the carrying out of one's occupation or profession: *Words are the tools of our trade.* "

([78], The American Heritage Dictionary of English Language, 2000)

"Attrezzo: [..] s. m. **1.** Ciascuno degli utensili o strumenti occorrenti ad una determinata attività: *gli a. del falegname; a. teatrali; a. navali* […] ."

([27], Devoto et al., pp206, 1971)

"Arnese: [..] s. m. **1.** Strumento o utensile di un'arte o mestiere […] ."

([27], Devoto et al., pp206, 1971)

11.5. Uses and usage

"Usage: act or mode of using, treatment; practice; custom […]"

([23], The Chambers Dictionary, 1998)

"Usage *n.* act or a manner of using; constant use, custom, or habit. […] control, employment, management, operation, running, treatment […] conventions, custom, form, habit, method, mode, practice, procedure, regime, routine, rule, tradition."

([24], Collins, pp. 652, 1994)

"Use: the act of using; the state or fact of being used; an advantageous purpose for which a thing can be used […]"

([23], The Chambers Dictionary, 1998)

"Use […] *n.* using or being used; ability or permission to use; usefulness or advantage; purpose for which something is used. […] application, employment, exsercise, handling, operation, practice, service, usage […]. Advantage, application, avail, benefit, good, help, point, profit, service, value, worth […]. Custom, habit, practice."

([24], Collins, pp. 652-653, 1994)

"Uso: m. Acción de utilizar o valerse de algo. […]"

([36], Garcia-Pelayo y Gross, pp. 604, 1983)

"Use: to put some purpose; to avail oneself; to employ habitually […]"

([23], The Chambers Dictionary, 1998)

"Use *[…] v*. put into service or action; behave towards in a particular way, usu. Selfishly; consume or expend. […] apply, employ, exercise, operate, ply, practice, utilize, wield, work […]. Exploit, handle, manipulate, treat […]. Consume, exhaust, expend, run through, spend, waste"

([24], Collins, pp. 652, 1994)

"Utilizar: v. t. Emplear, servirse de."

([36], Garcia-Pelayo y Gross, pp. 605, 1983)

11.6. Learning and Teaching

"Learn v. **learning, learnt** or **learned.** gain skill or knowledge by study, practice, or teaching; memorize (something); find out or discover. [..] 1. Acquire, attain, grasp, imbibe, master, pick up 2. Get off, pat, learn by heart, memorize 3. Detect, discern, discover, find out, gain, gather, hear, understand."

([24], Collins, pp. 353, 1994)

"Teach v. **teaching, taught** or **learned.** Tell or show (someone) how to do something; cause to learn or understand; give lessons (in a subject). […] advise, coach, direct, drill, educate, enlighten, guide, impart, implant, inculcate, inform, instil, instruct, school, show, train, tutor."

([24], Collins, pp. 614, 1994)

12. Appendix: Review on symbolic manipulators to teach symbolic manipulation

In this appendix, for the purposes of this thesis, I report a slightly modified version of a document that I presented at the "Working Group on Technological Environments" of the 12th ICMI Study Conference "The Future of the Teaching and Learning of Algebra", which was held in Mealbourne (Australia), December 9-14, 2001.

Each member of the working group prepared a research brief that was circulated among Working Group participants; the briefs, and the discussion of the working group, contributed to the realisation of a chapter ([46] Kieran and Yerushalmy, in press) dedicated to technological environments, in the 12th ICMI Study book ([46] Kieran and Yerushalmy, in press).

12.1. Introduction

Research literature includes an enormous quantity of papers concerning the use of technology in mathematics education and the range of papers remains vast even if we restrict to the teaching of algebra. Thus I am going to describe the research criterions that I used in order to help the reader understanding what I may have left out consciously or unconsciously.

The aim of this review was to identify research papers on a quite peculiar subject, symbolic manipulation, in particular I aimed at individuating research papers referring to software designed specifically to be used to teach symbolic manipulation. As a consequence I had two paths to follow:

4. Research for material (papers, pieces of software, demos, reviews) concerning the existing technology in order to find out which software have been designed for such a specific aim.
5. Look for research material (conference reports, books, journal papers) concerning the use of symbolic manipulators specifically designed to be used to teach symbolic manipulation.

From a first look at web archives (for instance Kluwer Academic web archive at http://www.wkap.nl) revealed that there are almost no relevant papers for 2, thus I enlarged my objective to:

6. Look for research material concerning the use of symbolic manipulators in the teaching of symbolic manipulation.

In the following I will first describe what I found out for 1. And then for 3.. In both cases I will first discuss how I searched for material, and then I will describe the obtained results, thus the reader may chose to skip some paragraphs and jump to the results.

12.2. The existing technology concerning symbolic manipulation

A quick look at the presentations of most famous computer algebra systems suggested me that actually almost no software seemed to be designed to introduce pupils to symbolic manipulation. As a consequence I shifted my attention to analyse whether a software could be classified as a symbolic manipulator or not. I personally reviewed a few software and classified them, in the luckiest cases I could test the software or a demo, in other cases I red their presentations on dedicated web pages.

12.2.1. What do we mean with *symbolic manipulator*?

For the aim of this review I used the following definition:

a symbolic manipulator (or calculator) is an environment providing instruments in order to operate on representations of algebraic expressions transforming them following mathematical rules.

Such a definition clearly excludes programming languages such as LOGO and PASCAL, while it includes for instance Mathematica, DERIVE, MAPLE, etc.

12.2.2. What is the objective of a software?

It is quite hard to find out what a software have been designed for, the only explicit information we have depends on how the software is presented by its creators and distributors. In this review, given an objective **O** and a software **S**, I considered **O** to be an objective of **S** iff one of the following conditions is verifued:

7. Its been declared to be so by the designers, or the creators, or the producers, or the distributors, or the maintainers.
8. It is distributed together with a package (CD, or booklet or whatever) describing how the software **S** could be used aiming at objective **O**. In case **O** is an educational objective the package should present an educational approach to **O** based on **S**.

In this review I used this criterion to analyse the available software.

12.2.3. What about the available software?

When we try a generic search on the internet for words such as *symbolic*, *manipulator*, *calculator*, *computer algebra system* etc. we find a huge quantity of different software packages. As a consequence it is impossible to classify all of them, thus I analysed a few in order to find out whether the software has been designed to teach symbolic manipulation ("teach" in the below table) or to be used to perform symbolic manipulation ("perform" in the below table). I left unfilled *objective* cells where I did not find information concerning the original objective of the software.

Software	Objective: teach vs perform
Alged	Perform
AMP	Perform
Derive	Perform
L'Algebrista	Teach
Maple	Perform
Mathematica	Perform
Mathomatic	Perform
Milo	
Symbolic Math Guide for T-91 & T-82	Teach
Symbmath	Perform
WICAT	Teach
EXPRESSIONS	Teach
RESOLVER	Teach
Theorist	

12.2.4. Information or promotional resources concerning software

The number of material describing or promoting specific software is very high, as high is the number of software. Here I am going to list only a few concerning the most popular pieces of technology.

Algebrator:

Algebrator – an Algebra Problem Solver for Students and Teachers. Available at **http://www.softmath.com/home-fr.htm** including an online demo and the below listed reviews.

A review of the software: *Technology*. Mathematics Teacher, Volume 93 n. 2, February 2000. National Council of Teachers of Mathematics.

A review of the software: *Software, focus on math*. The Journal, vol. 26 n. 10, May 1999.

DERIVE*:*

Kutzler (1995 and 1999)*. The author is the actual promoter of DERIVE in Europe, a critique on his discourse can be found in Lagrange*

Texas Instruments (2001): Derive™ 5 The Mathematical Assistant for Your PC. *Available at* **http://education.ti.com**.

Barozzi & Cappuccio (1997).

http://www.derive.com

http://www.kutzler.com

MACSYMA:

Martin W.A.,Fateman R.J. (1971): *The MACSYMA system*. **Proceedings of the second symposium on Symbolic and algebraic manipulation.**

Moses J. (1979): *The MACSYMA system for formula manipulation.*.**Proceedings of the APL Quote Quad conference part 1**.

Mathematica:

http://www.wolfram.com

Matlab:

Symbolic Math Toolbox (For Use with MATLAB). Copyright 1993-1998 by The MathWorks, Inc. available at **http://www.mathworks.com**.

MuPad*:*

Postel (1999) explains how the software can be used as a tool, a tutor and a tutee within school mathematics activities.

SureMath:

Grandgenett N. (1995): *Review of SureMath*. Mathematics and Computer Education, Vol. 29, No. 3, Fall 1995. Available at htttp://www.suremath.com

T-91 & T-82 and Symbolic Math Guide for T-91 & T-82:

Child (2000) presents the environment and gives some hints of how it could be used for educational purposes. The environment presents transparent commands to do symbolic manipulation step by step, in the paper it is discussed the case of exponential commands.

Texas Instruments (2001) TI-89/TI-92 Plus Symbolic Math Guide (guided tour). Available at **http://education.ti.com** Texas Instruments: Symbolic Math Guide A Concept APP for the TI-89 and TI-92 Plus. Available at **http://education.ti.com**.

Kutzler (1996).

12.3. Research on symbolic manipulators in the teaching/learning of symbolic manipulation

Many research papers concerning the teaching of symbolic manipulation do not even mention the expression "symbolic manipulation". Thus I needed to set up some criterions to individuate papers concerning such subject. I do not intend to give my own definition of "symbolic manipulation", I just list some aspects that have been taken into account in existing literature (e.g. Kieran 1998, Lagrange 2000, Mariotti Cerulli):

- Transformations of expressions
- transformations of equations
- equivalence relationships between expressions
- equivalence relationships between equations

- structure of expressions and equations

For this review I took into account only educational research resources involving any such items and involving the use of computers and calculators.

In order to have a continue story line of past and recent research I reviewed the proceedings of PME conferences from 1987 to 2001. In particular I followed indications contained on a survey of research reports concerning algebra within PME (Malara 1997).

The other main source I used is the *Kluwer Academic* on-line archive (http://www.wkap.nl) where I performed searches on the whole archive and on the specific archives of the following papers: *Journal of Mathematics Teacher Education*, *Educational Studies in Mathematics*; *Science & Education*; *International Journal of Computers for Mathematical Learning*; *International Journal of Technology and Design Education*; *Education and Information Technologies*; *Journal of Science Education and Technology*. I searched for combinations of the following words: symbolic; manipulation; manipulator; calculator; calculation; CAS; computer; algebra; software; mathematics.

I also did a generic internet search using the Google search engine (http://www.google.com) using the same key words.

12.3.1. Some relevant contributes in chronological order

1987. *Lesh and Herre* (PME[52]) focus on some results of computer based activities used to illustrates Dienes' instructional principles. The reported examples are based on the software SAM (by WICAT) "developed to enable students to write graph, transform, and solve algebraic expressions and equation" (Lesh, Herre 1987). In particular they expose an example concerning the resolution of second grade equations; in this case the software is used to transform the equation and to show at the same time graphic representations of the involved expressions. For instance, given the equation A[x]=B[x] the software plots y=A[x] and y=B[x]. When the equation is transformed into A[x]+C[x]=B[x]+C[x] the software plots the new functions y=A[x]+C[x] and y=B[x]+C[x]; this happens for any transformation rule applied to the equation. As a consequence it is possible to see how the set of solutions of the equation remains constant when the equation is transformed.

Thompson and Thompson (PME) present a study that took place over nine consecutive weekdays concerning an attempt to overcome students difficulties related to structures of algebraic expressions. The study is based on the use of EXPRESSIONS a "special computer program […] that enabled students to manipulate expressions, but which constrained them to acting on expressions only through their structure". The software represents expressions as trees and the use can operate on expressions by clicking on the action to do (for instance distribute, commute, etc.) and then clicking on the head of the branch of the represented expression that has to be transformed. The proposed activities consisted in transforming a given expression into another given expression. This was done both with numeric expressions and literal expressions. In particular we may observe that the software allowed step by step transformations and included transformation principles based on field properties, such as the properties of neutral elements. The authors present a brief analysis of students errors and conclude that attention to expression structure seems to be important. Furthermore they observe that pupils had no problem approaching manipulation with letters as such an activity seemed to base more one the structure of the expressions then on the nature of its terms. Finally they observe that few students spontaneously built expression trees to facilitate themselves evaluating them, this seems to be remarkable as the students had never seen an expression tree before.

1989. *Yerushalmy* (PME) reports a research aiming at studying a specific kind of difficulty encountered by students when transforming expressions: the absence of meaningful feedback. The

[52] Conference of the International Group for the Psychology of Mathematics Education.

researcher uses a software called RESOLVER (designed by Schwartz and Yerushalmy) that represents graphically expression transformations in the following way: the user enters an expression, then the user enters another expression obtained transforming (not in the computer) the original expression; the computer plots the graphic of the two expressions and the graphic of the difference between them, if the transformation is correct then such difference graphic would correspond to $y=0$. The user can go on inserting transformed expressions and at each step the software will plot the mentioned graphics. The author does not mention it, but it seems to be reasonable to imagine that the software works only with expressions with the variable x. The author presents some results of a couple of tests aiming at studying how students used the visual feedback; students are given expressions to be simplified.

In the introduction the author refers to some older researches on the use of computer for symbolic manipulation: “A Computer’s uses range from a tutor which direct students to carry the right simplification (Brown 1985), through computerized tools which direct students to understand the deep structure of algebraic expressions (Thompson 1987) to the use of programs that could carry symbolic transformations for the user such as MuMath (Fey 1984, Heid 1988)”.

1991. *Yerushalmy* (PME). “This study examined the effect of graphic representation of algebraic expressions on performance of tasks involving tranformations” (Yerushalmy 1991). The research follows up from previous 1989 PME report (Yerushalmy 1989), but in this case the experiment includes a teaching intervention where functions and transformations of algebraic expressions are introduced using single and multiple representation. The used software are *The Function Analyzer* (Schwartz & Yerushalmy 1989), *The Function Supposer* (*Analyzer* (Schwartz & Yerushalmy 1989) and *Transformer*.

1994. *Kieran* (PME) presents a project based on a functional approach to introduce students to algebra. To crate meaning for algebraic expressions two approaches are proposed, a process-oriented one and an object-oriented one. The first is based on a the software CARAPACE (Boileau & Garçon, 1987) in order to write algorithms to help building expressions from word problems; the object-oriented one is based on “Math Connections: Algebra II” (Rosemberg 1992), a software allowing working with graphs, tabulars and algebraic representations. Meaning for algebraic manipulations is created by comparing expressions using the mentioned software. Expressions are considered to be equivalent if they have the same graph or the same table of values. The study of equivalent expressions lead to the introduction of the main properties of algebraic expressions (associative, distributive, commutative).

1996. *Auricchio et al* (in Italian) give a detailed comparison of the computer programs DERIVE, MILO and Theorist from an educational point of view. The authors try to point out and compare those aspects of the interface that may help learning processes.

1999. *Yerushalmy* in a paper concerning more general issues discusses the special case of equivalent equations and transformations between them. Two technological approaches to the problem are described (with references to past researches) and discussed and a third one is proposed. The central node seems to be how the equivalence between equations is represented.

2001. *Mariotti and Cerulli* (PME). The authors discuss some results of a long term project concerning the introduction of students to symbolic manipulation within an axiomatic approach. The study is based on a symbol-manipulation microworld created by the authors. The paper concerns aspects of the theory of semiotic mediation in relation to the use of the specific software. The microworld presents strong analogies to the one used by Thompson and Thompson (1987), but the research aims at realizing an educational approach to algebra viewed as an axiomatic theory (Cerulli and Mariotti in press). Activities consist mainly on proofs of equivalencies between expressions; proofs consist on transformations bymeans of axioms (distributive, associative, commutative). The author aims also at intorducing students to the idea of theory and proof within a theory.

*Bouhineau, Nicaud, Pavard, Sander***.** Authors' abstract: "This paper describes the design principles of a microworld devoted to the manipulation of algebraic expressions. This microworld contains an advanced editor with classical actions and direct manipulation. Most of the actions are available in two or three modes; the three action modes are: a text mode that manipulate characters, a structure mode that takes care of the algebraic structure of the expressions, and an equivalence mode that takes into account the equivalence between the expressions. The microworld also allows to represent reasoning trees. The equivalence of the expressions built by the student is evaluated and the student is informed of the result. The paper also describes the current state of implementation of the microworld that will lead to a prototype available in February 2001."

12.3.2. Specially designed symbol-manipulation microworlds

The researches concerning microworlds specially created for learning/teaching symbolic manipulation that I found are the following ones: Lesh and Herre 1987; Thompson and Thompson 1987); Mariotti and Cerulli (2001).

12.3.3. Manipulation of expressions (or equations) preserving equivalencies

We find many researches focusing such a nodal question within symbolic manipulation. The followed approaches are mainly two:

- *multirepresentational approach*: equivalence between expressions (or equations) is rendered by using two or more representations (symbolic and graphic representations, and tables of values). The to transformation rules, that correspond to algebraic principles, are viewed as transformations that keep some invariant in the multiple representations of expressions (or equations). In this class we find Lesh and Herre 1987 (equations), Yerushalmy 1989 (expressions), Kieran 1994 (expressions), Yerushalmy 1999.

- *Structured manipulation*: the microworld offers commands that transform expression (or equations) operating on their structures and preserving equivalencies. Here we find Thompson and Thompson 1987, Mariotti and Cerulli (2001), Bouhineau et al. 2001.

12.3.4. Study of difficulties or intervention?

From a very general research objectives point of view we may split papers into two classes:

- papers concerning technological environments designed and/or used to study students difficulties: Yerushalmy 1989,

- papers concerning design and/or use of technological environments for teaching interventions: Lesh and Herre 1987, Thompson and Thompson 1987, Kieran 1984, Yerushalmy 1999, Cerulli Mariotti, Bouhineau et al. 2001.

12.3.5. Related but unchecked papers

There are a few papers that I could not access to but that might be related to the subject. I found references of such papers, and sometimes abstracts, using the following resources:

- Proceedings of Annual Conferences on Technology in Collegiate Mathematics (ICTCM) available at **http://archives.math.utk.edu**

- Web site of the Inst. De Recherche sur L'Enseignement des Mathematiques (IREM) **http://www.univ-montp2.fr/~irem**

- *Zentralblatt für Didaktik der Mathematik* (International Reviews on Mathematical Education).

For some of them I can give some indications on the contents:

Aczel 1998: Considers the contribution of computer software to understanding processes of solving equations.

Mainini 1998: discussion on computational skills concerning those calculations that are now done by computers.

Tynan et al. 1998: discussion of performance of pupils using CAS in symbol manipulation.

Frenc 1999: suggests how powerful manipulators may influence educational approaches to symbolic manipulation.

Gage 1999: presents a theorem prover that solves equations step by step.

Henrich 1999: influence of graphic calculators on pupils' computational skills.

Drijvers 2000: Experience of CAS for equations. Steps to develop functional use of mathematics tools.

Below is the whole list of such related papers:

Aczel J., Tilley D. (1998): *Algebra: rebalancing the equation*. Micromath (Summer 1998), v. 14(2), pg. 11-13.

Drijvers P., Van Herwaarden O. (2000): *Instrumentatie van ICT-gereedschap: algebranet computer-alggebra*. Nieuwe Wiskrant (sep 2000), v.20(1), pg. 38-43.

Fauvre C., Fontana J., Nogues M. (2000): *Calculatrices symboliques et algebre*. Montpellier-2 Univ.. Inst. De Recherche sur L'Enseignement des Mathematiques (IREM). L'algebre au lycée et au collège. Actes Journees de formation de formateurs. Boisseron (France).

Frenc D. (1999): *Factorizing with TI-92*. Mathematics in school (Harlow) (Jan 1999), v. 28(1), pg. 30-34.

Gage J. (1999): *Using the graphic calculator to teach algebra in lower secondary*. Proceedings of the 4th International Conference on Technology in Mathematics Teaching (ICMT 4). Maull W., Sharp J. (eds).

Guichard J.P. (1999): *L'algèbre au lyceé et au collège*. Actes des journées de formation des formateurs. Publication de l'institut de recherche sur l'enseignement des Mahématiques, Boisseron.

Grahm A., Thomas M. (1999): *A graphic calculator approach to algebra*. Mathematics Teaching (Jun 1999), n.167, pg. 34-37.

Henrich R. (1999): *Erziehen wir durch Verwendung grafikfarhiger Tescherechrer zu' Knopfchendruckern"*. Mathematik in der Schule (Mar-Apr 1999), v. 37(2), p. 107-110.

Horwitz A. (1995): *Using MATHEMATICA to Prove and Animate a Property of Cubic Polynomials*. Electronic Proceedings of the Eighth Annual Conference on Technology in Collegiate Mathematics, 8-C35. Huston. Available at http://archives.math.utk.edu.

Jurkovic (1986): *Edusym—educational symbolic manipulator on a microcomputer*. Proceedings of the symposium on Symbolic and algebraic computation . 154 - 156 Series-Proceeding-Article. ACM Press New York, NY, USA.

Lehmann, E. (1999): *Terme im Mathematikunterricht, unter Verwendung von Computergrafik und Computeralgebra*. Schroedel. Hannover (Germany).

Lichtenbergar F. (1984): *Self-explanatory Symbolic computation for Math Education*, SIGSAM Bulletin, Vol 18.

Mainini G. (1998): *Calcolatrice numerica e programmi di elaborazione simbolica*. Bollettino dei docenti di Matematica (Dec 1998), n.37, pg. 75-78.

Moses J. (1971).: *Algebraic simplification a guide for the perplexed*. Proceedings of the second symposium on Symbolic and algebraic manipulation, p.282-304, March 23-25, Los Angeles, California, United States

Stout D.R., Yun D.Y.Y. (1980): *Symbolic Mathematical Ccmputation*. Encyclopedia of Cc~puter Science and Technology, Vol 15.

Thomas M., Hall D. (1998): *A computer environment to encourage versatile understanding of algebraic equations*. Proceedings of the 21 annual conference of MERGA Inc., pg 605-612. Kanes C., Goo M., Warren E. (eds.). Gold Coast (Australia).

Tynan D., Asp G. (1998): *Exploring the impact of CAS in early algebra*. Proceedings of the 21 annual conference of MERGA Inc., pg 621-628. Kanes C., Goo M., Warren E. (eds.). Gold Coast (Australia).

Von Wright J. (1998): *Learning mathematics with a theorem prover*. Proceedings of the 4th International Conference on Technology in Mathematics Teaching (ICMT 4). Maull W., Sharp J. (eds).

12.4. Appendix references

Auricchio V., Dettori G., Lemut E. (1996): *Uso di software per la manipolazione algebrica nella scuola secondaria superiore*. Rapporto tecnico n. 10/96. Instituto per la Matematica Applicata – CNR, Genova.

Barozzi G.C., Cappuccio S. (1997): *Le calcolatrici grafiche nell'insegnamento della matematica*. Pitagora Editrice Bologna.

Boileau A., Garçon M. (1987): *CARAPACE* [Computer Program]. Montréal, Canada: Université du Québec à Montréal, Département de Mathématiques.

Bouhineau D., Nicaud J.F., Pavard X., Sander E. (2001): *Un micromonde pour aider les élèves à apprendre l'algèbre.* ATIEF – Sciences et Techniques Educatives, vol. 8. Rédactrice en chef Monique GRANDBASTIEN, Institut National Agronomique Paris-Grignon.

Brown J.S. (1983): *Process versus product: A perspective on tools for communal and informal electronic learning*. In Education in the Electronic Age, Report from the Learning Lab, WNET, NY.

Child D. (2001): *The symbolic math guide and the laws of exponents*. Micromath, Vol. 17 n. 2, Summer 2001. Edited by Wright D and Edwuards J.A.

Cerulli M., Mariotti M.A. (2000): *A symbolic manipulator to introduce pupils to algebra theory*. Proceedings of Workshop W6 "Learning Algebra with the Computer, a Transdisciplinary Workshop", Pg. 41-48. ITS'2000, Montreal.

Cerulli M., Mariotti M.A. (2001): *L'Algebrista: a microworld for symbolic manipulation*. Proceedings of the 12th ICMI Study Conference "The Future of the Teaching and Learning of Algebra". Edited by Helen Chick, Kaye Stacey, Jill Vincent & John Vincent. The University of Melbourne, Australia.

Cerulli M., Mariotti M.A. (in press): *L'Algebrista: a microworld for teaching and learning algebra* (title to be translated in french). ATIEF – Sciences et Techniques Educatives "Numéro spécial : Apprentissage de l'algèbre avec l'ordinateur". Rédactrice en chef Monique GRANDBASTIEN, Institut National Agronomique Paris-Grignon.

Fey J, (1984): *Computing and Mathematics: The Impact on Secondary School Curricula*. Reston, Virginia: National Council of Teachers of Mathematics.

Gafni, R. (1996): *Using multiple representations of function to improve semantics understanding of algebraic expressions and equations*. PhD dissertation (in Hebrew). Department of Mathematics Education, Faculty of Education, University of Haifa.

Heid K.M. (1988): *Resequencing Skills and Concepts in Applied Calculus Using the Computer as a Tool*. In Journal for Researhc in Mathematics Educational Vol. 19, No. 1, 3-25.

Heid, M. K. (1995): *Algebra in a technological world*. National Council of Teachers of Mathematics (NCTM), *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: NCTM.

Hillel, J. (1993): *Computer algebra systems as cognitive technologies: Implication for the practice of mathematics education*. In C. Keitel and K. Ruthven (Eds), *Learning from Computers: Mathematics Education and Technology* (pp. 18–47). Berlin Heidelberg: Springer-Verlag.

Kaput, J. and Roschelle, J. (1996): *SimCalc*. Project Report, SimCalc Project.

Kieran C., (1994): *A Functional Approach To The Introduction Of Algebra – Some Pros And Cons*. Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education "PME XVIII". Edited by Da Ponte J.P. and Matos J.F.. Lisbon, Portugal.

Kieran C., (1988): *Learning the structure of algebraic expressions and equations*. Proceedings of the twelfth International Conference for the Psychology of Mathematics Education "PME XII". Edited by A. Barbás. Veszprém, Hungary.

Kutzler B., (1995): I*mproving Mathematics Teaching with DERIVE*. Bromley:Chartwell-Bratt, 185 pages, ISBN 0-86238-422-2 (available in the US from west@trib.com, in the UK from **info@ChartwellYorke.com**).

Kutzler B., (1996): *Introduction to the TI-92 (Handheld Computer Algebra)*. Hagenberg:bk teachware, 184 pages, ISBN 3-901769-02-1

Kutzler B., (1999): *The Algebraic Calculator as a Pedagogical Tool for Teaching Mathematics*. In: Laughbaum E.D. (ed.): Hand-Held Technology in Mathematics and Science Education: A Collection of Papers, Teachers Teaching with Technology Short Course Program @ The Ohio State University, pp. 98-109. Also: **http://www.kutzler.com**.

Lagrange J.B. (2000): *Document pour l'Habilitation à Diriger les Recherches.* Université Paris VII 28 janvier 2000. Available at http://www.maths.univ-rennes1.fr/~lagrange/

Lagrange J.B. (1999): *Complex calculators in the classroom: theoretical and practical reflections on teaching pre-calculus*. International Journal of Computers for Mathematical Learning, vol. 4, pg. 59-81. Kluwer Academic Publishers. Printed in the Netherlands.

Lesh R., Herre J (1987): *Dienes revisited: Multiple embodiments in computer environments*. Proceedings of the Eleventh International Conference "Psychology of Mathematics Education PME – XI". Vol. I, pg. 211-220. Edited by J.C. Bergeron, N. Herscovics, C. Kieran. Montreal.

Little R. E. (1994): *Using A Calculator to do Mathematics: Do You Want a Map or just Directions?*. Electronic Proceedings of the Seventh Annual Conference on Technology in Collegiate Mathematics. *Editors:* Przemyslaw Bogacki (MANAGING EDITOR), Earl D. Fife, Larry Husch. Orlando.

Malara N. (1997): *Rassegna delle ricerche sui problemi di insegnamento-apprendimento dell'algebra in ambito PME*. Unpublished manuscript presented at Italian annual national conference on research in mathematics education "Seminario nazionale di ricerca in didattica della matematica". Pisa, 1997.

Mariotti M.A., Cerulli M. (2001): *Semiotic mediation for algebra teaching and learning*. Proceedings of the 25th conference of the International Group for the Psychology of Mathematics Education "PME 25@NL". Edited by Marja van den Heuvel-Panhuizen. Utrecht (The Netherlands).

Mayes R. (1997): *Current state of research into CAS in mathematics education*. In J. Berry and J.Monaghan (Eds), *The State of Computer Algebra in Mathematics Education* (pp. 171–189). Bromley: Chartwell-Bratt.

Postel F. (1999): *MuPAD as a Tool, Tutee and Tutor*. Primary talk at the Austrian Center for Didactics of Computer Algebra 1999 (ACDCA '99). Available at http://www.acdca.ac.at/kongress/goesing/g_postel.htm

Schwarts J.L., Yerushalmy M. (1989): *Visualizing Algebra: The Function Analyzer (computer software)*. Educational Development Center and Sunburst Communications, Pleasantville, NY.

Schwarts J.L., Yerushalmy M. (1990): *Visualizing Algebra: The Function Supposer (computer software)*. Educational Development Center and Sunburst Communications, Pleasantville, NY.

Schwartz, J. L. (1992): *Unsolving Equations and Inequalities*. Computer software. NY: Sunburst Communications.

Schwartz, J. L. and Yerushalmy, M. (1992): *Getting students to function in and with algebra*. In E. Dubinsky and G. Harel (Eds), *Learning the concept of function* (pp. 281–289). The Mathematical Association of America.

Schwartz, J. L. and Yerushalmy, M. (1995): *Calculus UnLimited*. Computer software. Tel-Aviv: Center for Educational Technology.

Rosemberg J. (1992): *Math Connections Algebra II* [Computer program]. Scotts Valley, CA: WINGS for Learning, Inc.

Thompson P.W., Thompson A.G. (1987): *Computer presentations of structure in algebra*. Proceedings of the Eleventh International Conference "Psychology of Mathematics Education PME – XI". Vol. I, pg. 248-254. Edited by J.C. Bergeron, N. Herscovics, C. Kieran. Montreal.

Yerushalmy M. (1989): *The use of graphs as visual interactive feedback while carrying out algebraic transformations*. Actes de la 13e conférence internationale "Psychology of Mathematics Education P.M.E. 13". Vol 3, pg. 252-260. Edité par Vergnaud G., Rogalski J., Artigue M.. Paris.

Yerushalmy M., Gafni R. (1991): *The effect of graphic representation: An experiment involving algebraic transformations*. Proceedings of the fifteenth PME conference (PME 15). Edited by Furinghetti F. Assisi.

Yerushalmy M. (1999): *Making exploration visible: on software design and school algebra curriculum*. International Journal of Computers for Mathematical Learning, vol. 4, pg. 169-189. Kluwer Academic Publishers. Printed in the Netherlands.

13. Examples of activities proposed to pupils

13.1. The preliminary test

T 2. *Observe the following writings, for each of them explain why you think it is correct, or why you think it is wrong.*

- $-(6-1)+3=-6+1+3$
- $17+(6+9)=(17+6)+9$
- $6+2\cdot(4\cdot5)=6+(2\cdot4)\cdot5$
- $3\cdot11+6-6=3\cdot11$
- $15+6\cdot4+19\cdot4+11=15+(6+19)\cdot4+11$
- $17=10+7$
- $5\cdot2+7=7+7$
- $8+9\cdot(3+2)-17=8+27+18-17$
- $10+7=5\cdot2+7$
- $3+6\cdot73+6\cdot8+13=3+6\cdot(73+8)+13$

T 3. *Write what you know concerning each of the following words and phrases, for instance you can write phrases containing them, or you can explain their meaning. You can also write examples.*

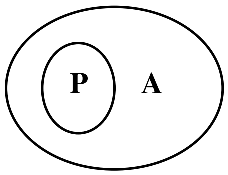
1. Evaluate
2. Result
3. Expression
4. Power
5. Associative property
6. Collect
7. Proprietà distributiva
8. Proprietà commutativa
9. Equazione
10. Espressione letterale
11. Prodotto notevole
12. Operazione
13. Somma algebrica di monomi
14. Monomio
15. Soluzione
16. Proprietà
17. Regola
18. Calcolo del valore di un'espressione
19. Formula
20. Disuguaglianza
21. Incognita
22. Uguale
23. Polinomio

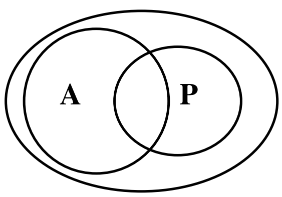
Which of the above phrases and words do you think are connected with each other? Why?

T 4. Which of the following writings may correspond to the phrase "*In the basket there are more apples then pears*":

- a + p

- a > p
- apples + pears
- p > a
- apple > pear
- p < a
-





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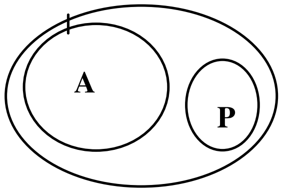
T 5. Which of the following phrases may correspond to the writing “*2a = p* “:



- For each pear there are two apples
- For each apple there are two pears
- There are double many pears then apples
- There are double many apples then pears

13.2. Numerical expressions

CL 1. Insert the following expression: **(3+5)*7-3*(2-8)**

- What happened to subtractions?

- Compute the result using the computation buttons.

CL 2. Compute, with the computation buttons: **7-5.**

Compute, with the computation buttons: **5-7.**

CL 3. Given the expression 3*(3+5). Compute the result and write a description of the followed computation procedure. Then, for the same expression, think of a different computation procedure, write and describe it, and verify that the results are the same.

Follows a class discussion on the equivalence the different computation procedures.

CL 4. Prove the following equivalence once transforming the leftmost expression, and once transforming the rightmost expressions: **3*4+3*7 == 3*(4+7)**

CS 1. We transformed two expression using the buttons of L'Algebrista, can you indicate, for each step, what button[53] we used and on which part of expression we applied them? Are all the steps correct?

2*3+2*5+2*(4+7)-(8+5)*2	3*(2*5)+(2*3)*5-60
$\boxed{\mathbf{2\cdot3+2\cdot5+2\cdot(4+7)+(-1)\cdot(8+5)\cdot2}}$	$\boxed{\mathbf{3\cdot(2\cdot5)+(2\cdot3)\cdot5+(-60)}}$
2·3+2·5+2·(4+7)+(-1)·(8+5)·2	3·(2·5)+(2·3)·5+(-60)
2·3+2·5+2·(4+7)+(2·(8+5))·(-1)	(3·2)·5+(2·3)·5+(-60)
2·3+2·(5+(4+7))+(2·(8+5))·(-1)	(3·2+2·3)·5+(-60)
2·3+2·(5+(4+7))+(2·8+5)·(-1)	(6+6)·5+(-60)
2·3+2·(5+(4+7))+((2·8)·(-1)+5·(-1))	(12)·5+(-60)
2·3+2·(5+(4+7))+(2·8)·(-1)+5·(-1)	60+(-60)
2·3+2·(5+4+7)+(2·8)·(-1)+5·(-1)	0

CS 2. Consider the following expressions:

7−2+6(3+4)+5·6

(3+4+5)·6+7−2

1) How would they look after being inserted in the program that we are using in the computer lab?

2) Check that they are equivalent computing their results

3) Prove that they are equivalent:

a) Transforming the first expression into the second one; explain each step with reference to the properties of the operations or to the buttons of the program.

b) Transforming the second expression into the first one; explain each step with reference to the properties of the operations or to the buttons of the program.

CL 5. Insert (in L'Algebrista) and compare the following expressions, if you think they are equivalent, prove it, otherwise check that they are not equivalent.

[53] Pupils were furnished with an image of the buttons available in L'Algebrista, Toeria 0 (see **Figure 2**).

`4*(7+11)+8*4+(3+17)*4+4*(23+2)==(7+11+17+3+8+23+2)*4`

Then answer the following questions:

1) What procedure did you followe? In particular, which buttons did you use?

2) Can you explain the functionning of each of the buttons that you used?

3) Which properties do they refer to?
4) Which is the button that you think it played the most important role in your script?

note: If it is not a problem for you, write youw ansers on the computer, below each question, otherwise write them in a sheet of paper.

CL 6. Insert (in L'Algebrista) and compare the following expressions, if you think they are equivalent, prove it, otherwise check that they are not equivalent:

`3(5*8)+8*11 == (3*5+11)*8`

Then answer the following questions:

1) What procedure did you followe? In particular, which buttons did you use?

2) Can you explain the functionning of each of the buttons that you used?

3) Which properties do they refer to?
4) Which is the button that you think it played the most important role in your script?

note: If it is not a problem for you, write youw ansers on the computer, below each question, otherwise write them in a sheet of paper.

CS 3. Consider the following three expressions:

`(3*(-8)+(3*7)*6-7*6)+(7+3)*6`

`7+3*6+3*(-8)+3*7*6-7*6`

`(-8+7*6)*3+(7*6+6*3-7*6)`

1) How would they look after being inserted in the program that we are using in the computer lab?

2) Compare the three expressions, are they all equivalent? If yes, prove it and explain each step with reference to the properties of the operations or to the buttons of the program.

3) What kinds of difficulties did you find?

4) Are you sure of all your answers, or is there anything that you would like to discuss in class? If yes, what is it about?

CS 4.

a) When an expression is inserted in the program, we noticed that subtractions are tranformed, can you say how? Why do you think the creator of the software decide to transform subtractions?

b) Why in some buttons of the program the operations are represented with the symbol "\" instead of with the usual simbols "+" for the sum and "." for the multiplication?

c) What do we mean when we say that two expressions are equivalent?

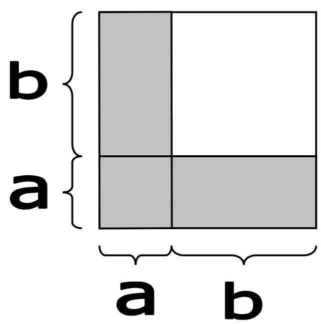
d) In class we distinguished between two words, *to prove* and *to check*, can you explain the meaning of each of them?

e) Write a scheme resuming the properties of the operations that we met till now, and write what we have learnt for each of them.

f) Do you think that the buttons of L'Algebrista are all the needed ones, or would you add some more button? If yes, explain how it should function.

13.3. Introduction of literal expressions

CL 7.



Discussing on the picture on the left, three friends found three different ways to compute the area of the part colored in grey:

Roberto proposes:

(a + b)*a + a*b

Valentina instead:

(a + b)*b - b*b + a*(a + b)

Finally Marco suggests:

b*b + (b + a)*a + (a - b)*b

1) Who do you think is right?

2) Why?

13.4. Proving equivalencies

CS 5. Consider the following expressions[54]:

a*a – b*b

(a – b)*(a + b) + 2*b*(a + b)

(a – b)*(a – b) + 2*(a – b)*b

1) Which of them do you think are equivalent?
2) Why? Can you prove it?
3) Analyse your proof and indicate, for each step, if you have used a theorem or an axiom (property)
4) Can you find a geometrical figure whose are can be computed by means of the above listed expressions?

CL 8. Consider the following expressions:

m + n +(2 + m)*n + (3 + m)*m + 3*(m + n) - m*(m + n)

7*m + 6*n

6*(m + n) + m*(n + 1) - n

[54] with this activity we introduce the words *axiom* and *theorem* as a way to distinguish between statement produced and proved in class by pupils, from given statements.

1) Which of them do you think are equivalent?
2) Why?

CS 6. Consider the following expressions:

$\mathbf{(m - n)^2}$

$\mathbf{2 \bullet (m + n) \bullet n + m^2 - 2 \bullet n \bullet n}$

$\mathbf{n \bullet m + (n^2 + m * m) - 3 \bullet m \bullet n}$

$\mathbf{m^2 - 2 \bullet m \bullet n + n^2}$

1) Which of them do you think are equivalent?
2) Why? Can you prove it?
3) Analyse your proof and indicate, for each step, if you have used a theorem or an axiom (property)
4) Can you find some numbers to subsitute to the letters **m** and **n** in the four expressions in order to obtain the same result from each of them?

   How many did you find?

   Do you think others can be found?

CL 9. Prove that[55] **13•m + m•17 == 30•m**

CL 10. Consider the following expressions:

(h - k)•(k + h)+(c - d)•(2•d - 2•c) - 8•c•d

8•c+(h - k)•(k + h) - 5•d

h•h - (c + d)•(2•d + 2•c)- k•k

h•h - k•k

1) Which of them do you think are equivalent?
2) Why? Can you prove it?
3) Analyse your proof and indicate, for each step, if you have used a theorem or an axiom (property)
4) If you substitute numbers to the letters **c** and **d**, you obtain new expressions containing only the letters **h** and **k**. Can you find numbers to substitute to **c** and **d** so that two non equivalent expressions become equivalent?

CS 7.

1. Prove that:
   a. **a – a == 0**
   b. **b•5 – 6•b + b == 0**
   c. **7•m + 2** is not equivalent to **3 + m•2**

[55] The symbol "==" in this case, according to the notation shared by the class, stands for "equivalent".

d. $x^2 - y^2 == (x - y)\bullet(x + y)$

e. **(3•a + m•2)** is not equivalent to $2^2\bullet(a^2 + m^2 + 3\bullet m\bullet a)$

5) Analyse each of the proofs you produced and indicate, for each step, if you have used a theorem or an axiom (property).

2. Considers the two expressions of 1.c.: can you find numbers to substitute to **m** so that the two expressions become equivalent?
3. Considers the two expressions of 1.e. can you find numbers to substitute to **a** so that the two expressions become equivalent?
4. Invent two literal expressions that are not equivalent, but that can become equivalent by substituting particular numbers to the letters.

13.5. Introducing new theorems with the Teorematore

CL 11. Prove that $(a - b)\bullet(a + b) = a^2 - b^2$

CL 12. Conjecture a formula for the expression $(a+b)^3$ and create a new button representing it. Are you sure that it is correct? Why?

CL 13. Prove the equivalence $(a + b)^3 == (a + b)\bullet(a^2 + b^2 + 2\bullet a\bullet b)$. Can you reduce the number of steps of your proof?

CL 14. Consider the expression $(a - b)^2$, conjecture a formula analogous to the one that we found for $(a + b)^2$ and compare the created buttons. Do you think both formulas are necessary? Or do you think one can be used to accomplish also the tasks of the other? And what do you think in the case of their buttons?

CS 8. Conjecture a formula for the expression $(a + b)^4$. Are you sure that it is correct? Why?

CS 9. Prove the equivalence $(a + b)^2\bullet(a - b) == (a^2 - b^2)\bullet(a + b)$. Can you reduce the number of steps of your proof?

13.6. Revising the work done

CL 15. Consider the following equivalencies, prove them if you think they are true, otherwise prove that they are false.

I. $a\bullet a = a^2$

II. $(-b)\bullet a = -(a\bullet b)$

III. $(-b)\bullet b = (-1)\bullet b^2$

IV. $-(b\bullet b) = (b\bullet b)\bullet(-1)$

V. $(-b)\bullet(-b) = b^2$

VI. $a\bullet(-b) + b\bullet a = a\bullet 0\bullet a$

VII. $a\bullet(-b) + b\bullet c = a\bullet 0\bullet c$

CL 16. Someone of you wrote that in order to prove that two expressions are equivalent (or not equivalent), he/she "*computed them algebraically*". What do you think he/she meant to say?

CS 10. Consider the following dialogue:

Valentina: *"To prove that two literal expressions are equivalent (or not equivalent), it is enough to substitute numbers to the letters."*

Marco: *"I think, instead, that it sufficies to transform one expression into the other, using the available axioms and theorems."*

Luigi: *"I think that you are both wrong, in fact if I cannot transform an expression into the other, how can I know if they are really different, or if it is me who is not able to prove their equivalence? Moreover, if ,substituting numbers to the letters, the two expressions lead to the same result, how can I be sure that ther isn't any number for which the results are different? "*.

What is your opinion? You can help yourself showing examples.

CL 17. During the activity we did in class last thursday, some of you wrote the following expressions, that have an odd result whatever is the number substituted to the letters:

a. $\mathbf{8 \bullet a + 1}$

b. $\mathbf{y^0 \bullet x^0}$

c. $\mathbf{x/x}$

Which odd numbers is it possibile to obtain?

Is it possibile to obtain the number **5** with any of these expressions?

Write an expression with which it is possibile to obtain all the odd numbers.

Write a similar expression for even numbers.

13.7. Factorisation and other activities propaedeutical to the introduction of equations

CL 18.

I. Prove that $\mathbf{2 \bullet a + 2}$ is divisible by **2**.

II. Prove that $\mathbf{3 \bullet a + 7 - 4}$ is divisible by $\mathbf{(a+1)}$

III. Prove that $\mathbf{a^2 - 4}$ is divisible by $\mathbf{(a+2)}$. Is it divisible by anything else?

IV. Prove that $\mathbf{n \bullet (m + 2) \bullet 3 \bullet (4 + n)}$ and $\mathbf{4 \bullet n + n^2}$ do not have any common factor.

CS 11.

- Prove that $\mathbf{b^2 + 4 \bullet c \bullet b + 2 \bullet (2 \bullet c^2)}$ is divisible by $\mathbf{(b+2 \bullet c)}$
- Prove that $\mathbf{a^2 + b^2 + 2 \bullet a \bullet b}$ and $\mathbf{a^2 - b^2}$ share some factors.

CS 12.

I. Write a literal expression using the letter **a** so that, substituting the number **2** to the letter **a** the result of the obtained numerical expression is **0**. *(If you want you can use also other letters)*

II. Try to substitute other numbers to the letter **a**, can you find other that lead to the result **0**.

III. Try now to build another expression, with the letter **a**, so that substituting **4** or **5** to **a** you get the result **0**.

CS 13.

I. Prove that $a^2 \bullet a^3 == a^5$

II. Prove that $\mathbf{a^b \bullet a^c == a^{(b+c)}}$

III. Prove that $\mathbf{a^{(3\bullet 2)} == (a^3)^2}$

IV. Prove that $\mathbf{a^{(b\bullet c)} == (a^b)^c}$

CL 19. Consider the number **2•3•4•6•7•9•10•13•17•27•31**

a. Can you say, without computing it, if it is divisible by **2**? And by **19**?
b. Can you say, without computing it, if it is a multiple of **3**? And of **8**? And of **13•7**?
c. For which other numbers is it divisible?
d. Is it a multiple of which other numbers?
e. Can you find a formula to express the fact that a generic number is divisible by **31**?
f. Can you find a formula to express the fact that a generic number is multiple of **31**?
g. Can you build a number which is divisible by 23 but that is not divibles by 3?
h. Can you build a number which is multiple of 36 but not of 4?
i. Build a literal expression so that, whatever is the number you substitute to the letters, the result is always odd.

CL 20. Consider the expression **2•(a+3)•6•(b-1)**

a. What numbers is it divisible by? Explain your reasoning.
b. What expressions is it multiple of? Explain your reasoning.
c. Build an expression that is divisible by **2**, but not by **3**
d. Build an expression that is multiple of **a+1**
e. Build two expressions that share the factor **3** but that are not equivalent.
f. Build an expression that is divisible by **3** and by **(a – b)**; how many of such expressions is it possible to build?

CS 14. <u>The truth game:</u> Luigi, Marco, Roberto and Valentina read on their book the expression: **(2•a + 4•b)•(a – 2•b + a – 2•b)**.

The teacher asks them what they can argue concernine such expression; the pupils answer:

Marco: I think that it is equivalent to **$2•(a+2•b)•(2•a – 4•b)$**

Valentina: I think you are wrong, in fact it is not divisible by **2** but it is divisible by **4**

Roberto: I think that it is equivalent to **a $4•a^2 + 4•b^2$**

Luigi: I believe, instead, that it is equuivalent to **$(2•a)^2 – (4•b)^2$**

What would you say?

CL 21. For each of the following statements explain why you think it is true or false.

a. The expressions **$9•3•5 + 7^2$** and **$9•3•5 – 5^3$** are different, but share **3** as a common factor

b. **$(a – b)•3•x + 7$** is multiple of either of **$(a – b)$** and of **3**

c. The two following expressions are different but share three factors

$(3•a – 4)•b^2•5•c•(a^2 – a)•4•a•3•b•2•c$

$(4•a – 3)•b^2•5•c•(a^2 – a)•2•b•4•c•5•a$

d. The two following expressions are different but share three factors

$a•b•c^2 + b^2$

$a^2•b + c + 3•a•b$

e. The following expression is divisible either by **3** and by **$(a – b)$**

$3•(x + y + z) + a•b•(a – b)$

CL 22. Factorize the following expressions (hitn: try to find all their divisors)

$a^2 – b^2 + a – b$
$b^2 + a + 2•a•b + b+ a^2$

CS 15. Consider the expression **$a + b +(3•a – 3•b)•2 + 3•b$**, which equivalent expression do you think it is convenient to transform it into, in the case that:

a. You have to compare it with another expression.

b. You have to prove that it is divisible by **2** and not by **3**.

c. You have to prove that it is divisible by **$a + b$**.

d. You have to substitute numbers to **a** and **b** to compute it, and you want the computation to be the easiest possible.

CL 23. Compare the following expressions

$(m + 2)•(m – 3)•m•(m – 4)•(m – 5)(2 – m – 2•m^2 + m^3)$

$(m – 1)•(m + 1)•(m – 2)•(34•m^2 – 120•m + 23•m^3 – 10•m^4 + m^5)(m – 6)$

$(m + 2)•(m – 3)•m•(m – 4)•(m – 5)•(m – 1)•(m + 1)•(m – 2)$

a. Are they equivalent or different? Why?

b. Factorize the three expressions

c. How many factors do they share?

d. Which numbers can you substitute to the letter **m** in order to get the result **0**.

13.8. A final test

FIRST PART

1. **The symbol "="**
   a. What do you think the symbol "=" means in algebra?
   b. Write examples using this symbol
2. **The word "equal"**
   a. What do you think the word "equal" means in algebra?
   b. Write examples using this word
3. **The symbol "equivalent"**
   a. What do you think the word "equivalent" means in algebra?
   b. Write examples using this word
4. **Other symbols and words**
   a. What relationship do you think there is between the symbol "=", the word "equal" and the wor "equivalent"? You can eventually show examples
   b. Write other symbols and words that you know, and you think are related to the symbol "=" or to the words "equal" and "equivalent". You can eventually show examples
5. **Algebraic expression**
   a. What do you think an algebraic expression is
   b. What do you think it can be done using an algebraic expression? You can eventually show examples
   c. What do you think it can be done using an expression in Mathematics? You can eventually show examples
   d. What relationship is there between numerical expressions and literal expressions? Can you do the same things using them?
6. **Axioms, theorems and proofs**
   a. What do you think an axiom is in algebra? What can it be used for? You can eventually show examples
   b. What do you think a theorem is in algebra? What can it be used for? You can eventually show examples
   c. What do you think a proof is in algebra? What can it be used for? You can eventually show examples
   d. What do you think a theory is in algebra? What can it be used for? You can eventually show examples

SECOND PART[56]

7. **L'Algebrista**
   a. Suppose you have to explain to someone what L'Algebrista is, what would you tell him/her?

[56] The second part is submitted to pupils only after they finished the first part.

b. What do you think L'Algebrsita can be used for? You can eventually show examples
c. Do you think L'Algebrista is ok how its is now, or would you change anything? You can eventually show examples
d. What aspect of L'Algebrista do you like most, and what aspect do you like less?
e. What relationship is there between the symbol "= =" of L'Algebrista, and the symbol "="?
f. Do you think that an expression of L'Algebrista, and an expressions written on paper, are equal or different? Why? You can eventually show examples
g. What elements of L'Algebrista correspond respectively to the axioms and the theorems of algebra?
h. Do you think that proving in L'Algebrista is the same as proving in paper and pencil? Do you think there are any differences? Why? You can eventually show examples
i. If you would have to choose to accmplish a task in L'Algebrista or in paper and pencil, what would you choose? Why?
j. Was L'Algebrista anyhow useful for you? Did you learn anything by using it?

14. Bibliography

[1] Arcavi, A.: 1995, *Teaching and Learning Algebra: Past, Present, and Future*. Journal of mathematical behaviour 14, 145-162.

[2] Arzarello, F., & Robutti, O. (2001). "From body motion to algebra through graphing". In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The future of the teaching and learning of algebra* (Proceedings of the 12th ICMI Study Conference), vol. 1, pp. 33-40. Melbourne: The University of Melbourne. Australia.

[3] Balacheff, N.: 2002, "The researcher epistemology: a deadlock from educational research on proof". Fou Lai Lin (ed.) 2002 *International Conference on Mathematics - "Understanding proving and proving to understand"*. Taipei: NSC and NTNU (pp. 23-44).

[4] Balacheff, N., Kaput, J.J.: 1996, "Computer-Based Learning Environments in Mathematics". In Bishop, A.J. et al., *International Handbook of mathematics education* (pp. 469 - 501). Kluwer Accademic Publishers. Dordrecht, Netherlands.

[5] Balacheff, N., Sutherland, R.: 1994, *"Epistemological domain of validity of microworlds, the case of Logo and Cabri-géomètre"*. In Lewis, R. & Mendelson P. (Eds.), Proceedings of the IFIP TC3/WG3.3: Lessons from learning (pp. 137-150). Amsterdam: North-Holland,.

[6] Bartolini Bussi M. G.(1996). *Mathematical Discussion and Perspective Drawing in Primary School*. In Educational Studies in Mathematics, 31 (1-2), 11-41.

[7] Bartolini Bussi M. G.: 1998, *Verbal Interaction in Mathematics Classroom: a Vygotskian Analysis*, in Steinbring H., Bartolini Bussi M. & Sierpinska A. (eds.), Language and Communication in the Mathematics Classroom, Reston VA: NCTM, 65-84.

[8] Bartolini Bussi M. G., & Mariotti M. A.: 1999, "Instruments for Perspective Drawing: Historic, Epistemological and Didactic Issues". In Goldschmidt G., Porter W. & Ozkar M. (Eds.), *Proc. of the 4th Int. Design Thinking Res. Symp. on Design Representation*, III 175-185, Massachusetts Institute of Technology & Technion - Israel Institute of Technology.

[9] Bednarz, N.: 2001, "*A Problem-Solving Approach to Algebra: Accounting for the reasoning and Notations Developed by Students*". Proceedings of the 12th ICMI Study Conference "The Future of the Teaching and Learning of Algebra", vol. 1, pp. 69-78. Edited by H. Chick, K. Stacey, Jill Vincent & John Vincent, held in December 9-14, 2001, at the University of Melbourne, Australia.

[10] Boero, P., Dapueto, C., Ferrari, P., Ferrero, E., Garuti, R., Lemut, E., Parenti, L., Scali, E. (1995). *Aspects of the Mathematics-Culture Relationship in Mathematics Teaching-Learning in Compulsory School*, In Proceedings of 19th PME Conference (pp. I-151-66). Recife, Brazil.

[11] Bottino, R. M.: 2001?, *Advanced Learning Environments: Changed Vies and Future Perspectives*. In "Computers and Education Towards an Interconnected Society", pp. 11-26. Edited by M. Ortega, J. Bravo. Kluwer Academic Publishers, Dordrecht/Boston/London.

[12] Bottino, R. M., Chiappini, G.: 2002, *Advanced Technology and Learning Environments: Their relationships within Arithmetic Problem-Solving Domain*. In "Handbook of Interantional Research in Mathematics Education", chapter 29, pp. 757-786. Edited by L. D. English. Lawrence Erlbaum Associates publishers, Mahwah, New Jersey.

[13] Bouhineau D., Nicaud J.F., Pavard X., Sander E. (2001): *Un micromonde pour aider les élèves à apprendre l'algèbre.* ATIEF – Sciences et Techniques Educatives, vol. 8. Rédactrice en chef Monique GRANDBASTIEN, Institut National Agronomique Paris-Grignon.

[14] Brown, L., Coles, A.: 2001 *"Natural Algebraic Activity"*. Proceedings of the 12th ICMI Study Conference "The Future of the Teaching and Learning of Algebra", vol. 1, pp. 120-127. Edited by H. Chick, K. Stacey, Jill Vincent & John Vincent, held in December 9-14, 2001, at the University of Melbourne, Australia.

[15] Carpenter, T. P., Franke, M. L.: 2001, *"Developing Algebraic Reasoning in the Elementari School: Generalization and Proof"*. Proceedings of the 12th ICMI Study Conference "The Future of the Teaching and Learning of Algebra", vol. 1, pp. 155-162. Edited by H. Chick, K. Stacey, Jill Vincent & John Vincent, held in December 9-14, 2001, at the University of Melbourne, Australia.
[16] Cerulli M.: 1999, *Uso di manipolatori simbolici nell'introduzione all'algebra: realizzazione e sperimentazione di un prototipo*. Unpublished manuscript, Tesi di Laurea in Matematica, "Università degli Studi di Pisa", Pisa.
[17] Cerulli M.: 2002, Introducing pupils to theoretical thinking: the case of algebra. *Proceedings of CERME2, European Research in Mathematics Education II*, vol. 1 pg 139-151. Edited by Jarmila Novotna, Prague 2002.
[18] Cerulli M.: 2004, "Instruments of semiotic mediation in Algebra, an example". *CERME 3 European Research in Mathematics Eucation III, proceedings the 3rd Conference of the European society for Research in Mathematics Education*, 28 Feb – 3 Mar 2003, Bellaria, Italy.
[19] Cerulli, M., Mariotti, M.A.: 2000, A symbolic manipulator to introduce pupils to algebra theory. Proceedings of Workshop W6 "Learning Algebra with the Computer a Transdisciplinary Workshop". ITS'2000, Montreal.
[20] Cerulli, M., Mariotti, M. A.: 2001, *Arithmetic and algebra, continuity or cognitive break? The case of Francesca*. Proceedings of the 25th International Conference "Psychology of Mathematics Education PME25". Vol. II, pp. 225-232. Edited by Marja van den Heuvel-Panhuizen. Freudenthal Institute, Utrecht..
[21] Cerulli, M., Mariotti, M. A.: 2002, *L'Algebrista: un micromonde pour l'enseignement et l'apprentissage de l'algèbre*. In "Science et techniques éeducatives", vol. 9, *Logiciels pour l'apprentissage de l'algèbre*, pp. 149-170. Hermès Science Publications, Lavoiser, Paris.
[22] Cerulli, M., Mariotti, M. A.: 2003, *Building theories: working in a microworld and writing the mathematical notebook*. In "Proceedings of the 2003 Joint Meeting of PME and PMENA". Vol. II, pp. 181-188. Edited by Neil A. Pateman, Barbara J. Dougherty, Joseph Zilliox. CRDG, College of Education, University of Hawai'i, Honolulu, HI, USA.
[23] Chambers Harrap Publishers Ltd.: 1998, *The Chambers Dictionary*. Chambers Harrap Publishers Ltd. Edimburgh, UK.
[24] Collins: 1994, *Paperback Dictionary and Thesaurus*. HarperCollins Publishers, Glasgow.
[25] De Corte, E.: 1996, "Changing views of of computer supported learning environments for the acquisition of knowledge and thinking skills". In S. Vosniadou, E. De Corte, R. Glaser, & H. Mandl (eds.), *International perspectives on the design of technology-supported learning environments* (pp. 129-145). Hillsdale, NJ: Lawrence Erlbaum Associates.
[26] Dettori, G., Garuti, R., & Lemut, E. (2000). From arithmetic to algebraic thinking by using a spreadsheet. In R. Sutherland, T. Rojano, A. Bell, & R. C. Lins (Eds.), *Perspectives on school algebra* (pp. 191-207). Dordrecht, The Netherlands: Kluwer Academic.
[27] Devoto, G., Oli, G. C.: 1971, *Dizionario della Lingua Italiana*. Felice Le Monnier, Firenze.
[28] diSessa, A.: 1988, "Social Niches for Future Software". In A. diSessa, M. Gardener, J. Greeno, F. Reif, A. H. Schoenfeld, & E. Stage (Eds.), *Towards a Scientific Practice of Science Education*, Lawrence Erlbaum Associates. Hillsdale, New Jersey.
[29] diSessa, A.A. (1988). Knowledge in Pieces. In G. Forman & P. Pufall (Eds.), *Constructivism in the Computer Age* (pp. 49-70). Hillsdale, New Jersey: Lawrence Erlbaum Associates, Inc.
[30] diSessa, A.A. (1995). The Many Faces of a Computational Medium: Teaching the Mathematics of Motion. In A.A. diSessa, C. Hoyles, R. Noss, & L. Edwards (Eds.), *Computers for Exploratory Learning* (pp. 337-359). Berlin: Springer–Verlag.

[31] Dugdale, S. (1993). Functions and graphs - perspectives on students' thinking. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 101-130). Hillsdale, NJ: Lawrence Erlbaum.
[32] Duval R.: 1995, *Sémiosis et pensée humaine*. Peter Lang, Berne.
[33] Eco, U.: 1973, *Il segno*. ISEDI, Istituto Editoriale Internazionale, Milano.
[34] Eco, U.: 1975, *Trattato di Semiotica Generale*. "Studi Bompiani, Il campo semiotico *a cura di Umberto Eco*". Gruppo Editoriale Fabbri, Bombiani, Sonzogno, Etas S.p.A., Milano.
[35] Eisember, M.: 1995, "Creating Software Applications for Children: Some thoughtes About Design". In A. diSessa, C. Hoyles, & R. Noss (Eds.), *Computers and Exploratory Learning*, pp. 175-196. NATO, Berlin.
[36] Garcia-Pelayo y Gross, R.: 1983, *Diccionario Larousse del Español moderno*. Libraire Larousse, New York, U.S.A..
[37] Guin., D., Trouche, L.: 1999, *The complex process of converting tools into mathematical instruments: the case of calculators*. International Journal of Computers for Mathematical Learning, n. 3, pp. 195-227. Kluwer Academic Publishers, The Neathelands.
[38] Filloy, E., Rojano, T., & Rubio, G. (2000). Propositions concerning the resolution of arithmetical-algebra problems. In R. Sutherland, T. Rojano, A. Bell, & R. C. Lins (Eds.), *Perspectives on school algebra* (pp. 155-176). Dordrecht, The Netherlands: Kluwer Academic.
[39] Henry, V. J.: 2001, "An Examination of Educational Practices and Assumptions Regarding Algebra Instruction in the United States". In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The future of the teaching and learning of algebra* (Proceedings of the 12th ICMI Study Conference), vol 2, pp. 296-304. Melbourne: The University of Melbourne. Australia.
[40] Huntley, M. A., Rasmussen, C. L., Villarubi, R. S., Sangtong, J., & Fey, J. T. (2000). Effects of Standards-based mathematics education: A study of the Core-Plus Mathematics Project Algebra and Functions Strand. *Journal for Research in Mathematic Education, 31*, 328-361.
[41] Ilienkov, E.: 1977, *Dialectical Logic*. Progress Publishers, Moscow.
[42] Illich, I.: 1973, *Tools for Conviviality*. Calder and Boyars, London.
[43] Kaput, J.J, Blanton, M.: 2001, "Algebrafying the Elementary Mathematics Experience". In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The future of the teaching and learning of algebra* (Proceedings of the 12th ICMI Study Conference), vol 2, pp. 344-351. Melbourne: The University of Melbourne. Australia.
[44] Kieran, C.: 1992, *The learning and teaching of School Algebra*. In Handbook of Research on Mathematics Teaching and Learning. D. A. Grouws ed., N.C.T.M..
[45] Kieran, C.: 2003, *The transition from arithmetic to algebra*. In "Matématica educativa *Aspectos de la investigación actual*". Edited by Eugenio Filloy. Centro de Investigación y de Estudios Avanzados del IPN, Fondo de Cultura Económica, Mexico.
[46] Kieran, C., Yerushalmy, M. (in press): *Research on the role of technological environments in algebra learning and teaching.* To appear in the book of ICME study on algebra (Mealbourne 2001), chapter 5.
[47] Kuutti, K: 1996, "Activity Theory as a Potential Framework for Human-Computer Interaction Research". In Nardi, B. A. (ed.), *Context and Consciousness*, pp. 17-44. Cambridge MA: The MIT press.
[48] Lee, L.: 2001, "Early Algebra – but Which Algebra". In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The future of the teaching and learning of algebra* (Proceedings of the 12th ICMI Study Conference), vol 2, pp. 392-399. Melbourne: The University of Melbourne. Australia.
[49] Lee L. & Wheeler D.: 1989, The arithmetic connection. *Educational Studies in Mathematics, 20,* 41-54.

[50] Malara, N. A: 1997, *Rassegna delle ricerche sui problemi di insegnamento-apprendimento dell'algebra in ambito PME*. Unpublished manuscript presented at the "Seminario Nazionale di Ricerca in Didattica della Matematica", 11-13 Dec 1997, Pisa.

[51] Mariotti, M. A.: 2002, *Influencies of technologies advances in students' math learning*. In "Handbook of Interantional Research in Mathematics Education", chapter 29, pp. 757-786. Edited by L. D. English. Lawrence Erlbaum Associates publishers, Mahwah, New Jersey.

[52] Mariotti M.A.: 2001, *Introduction to proof: the mediation of a dynamic software environment*. "Educational Studies in Mathematics", 44, pg 25-53, Kluwer Academic Publisher, Netherlands.

[53] Mariotti, M.A.: 1997, *Justifying and proving: figural and conceptual aspects*. In Hejny M. & Novotna J. (Eds.), Proceedings of the European Research Conference on Mathematical Education (pp. 21-27). Podebrady, Czech Republic.

[54] Mariotti, M.A.: 1996, *Costruzioni in geometria*, in "L'insegnamento della Matematica e delle Scienze Integrate", 19B, (3), 261 - 88.

[55] Mariotti, M.A. & Bartolini Bussi M.G.:1998, From drawing to construction: teachers mediation within the Cabri environment, in *Proceedings of the 22nd PME Conference,* Stellenbosh, pp. I- 180-95.

[56] Mariotti M.A., Cerulli M.(2003): *"Espressioni numeriche ed espressioni letterali: continuità o rottura?"*. In "La matematica e la sua didattica", n. 1 – 2003, pg 43-63. Pitagora Editrice.

[57] Mariotti M.A., et al.: 1997, Approaching geometry theorems in contexts: from history and epistemology to cognition, *Proceedings of the 21st PME Conference,* Lathi, pp. I- 180-95.

[58] Merriam-Webster: 2003, *Merriam-Webster Dictionay*. Merriam-Webster Incorporated, http://www.m-w.com/home.htm.

[59] Mokros, J. R., & Tinker, R. F. (1987). The impact of microcomputer-based labs on children's ability to interpret graphs. *Journal of Research in Science Teaching, 24,* 369-383.

[60] Nardi, B. A.: 1996, "Studying context: a comparison of Activity Theory, Situade Action Models, and Distributed Cognition". In Nardi, B. A. (ed.), *Context and Consciousness*, pp. 69-102. Cambridge MA: The MIT press.

[61] Noss, R.: 2002, *Mathematical epistemologies at work*. For the Learning of Mathematics, vol 22, number 2, pp. 2-13. FLM Publishing Association. Kingston, Ontario, Canada.

[62] Noss, R., Hoyles, C.: 1996, "Windows on mathematical meanings learning cultures and computers". *Mathematics Education Library*, vol. 17. Kluwer Academic Publishers, Dordrecht/Boston/London.

[63] Papert, S. (1996). An Exploration in the Space of Mathematics Educations. *International Journal of Computers for Mathematical Learning, 1*(1), 95-123. Kluwer Academic Publishers. Printed in the Netherlands.

[64] Pratt, D.: 1998, *The construction of meanings in and for a stochastic domain of abstraction*. Unpublished doctoral dissertation, University of London, Institute of Education.

[65] Prodi G.: Matematica come scoperta, 1, (Prima Edizione). Casa editrice G. D'Anna, Messina-Firenze, 1975.

[66] Rabardel, P.: 1995, Les Hommes & Les Technologies (Aproche cognitive des instruments contemporains). Armand Colin Editeur, Paris.

[67] Radford, L., Furinghetti, F.: 2002, Historical Conceptual Developments and the Teaching of Mathematics: from Phylogenesis and Ontogenesis Theory to Classroom Practice. In "Handbook of Interantional Research in Mathematics Education", chapter 29, pp. 757-786. Edited by L. D. English. Lawrence Erlbaum Associates publishers, Mahwah, New Jersey.

[68] Radford, L.: 2003, "Gestures, Speech, and the Sprouting of Signs: A Semiotic-Cultural Approach to Students' Types of Generalization". In *Mathematical Thinking and Learning*, 5(1), 37-70. Lawrence Erlbaum Associates, Inc.

[69] Radford L., Demers S., Guzmàn J., Cerulli M.: 2003 "*Calculators, graphs, gestures and the production of meaning*". Vol. IV, pp. 55-62. Edited by Neil A. Pateman, Barbara J.

Dougherty, Joseph Zilliox. CRDG, College of Education, University of Hawai'i, Honolulu, HI, USA.
[70] Salomon, G.: 1996, "Computers as a trigger for change". In S. Vosniadou, E. De Corte, R. Glaser, & H. Mandl (eds.), *International perspectives on the design of technology-supported learning environments* (pp. 363-377). Hillsdale, NJ: Lawrence Erlbaum Associates.
[71] Schwartz, J. L., & Yerushalmy, M. (1995). On the need for a bridging language for mathematical modeling. *For the Learning of Mathematics, 15*(2), 29-35.
[72] Sendov, B., & Sendova, E.: 1993, "East or West – GEOMLAND is Best, or Does the Answer Depend on the Angle?". In A. diSessa, C. Hoyles, & R. Noss (Eds.), *Computers and Exploratory Learning*, pp. 59-68. Springer Verlag, Berlin.
[73] Sfard A.: 1991, *On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin*. Educational Studies in Mathematics, 22 (pg. 1-36).
[74] Sfard A., Linchevski, L.: 1994, "The gains and the pittfalls of reification – the case of algebra". In *Educational Studies in Mathematics* 26, pp. 191-228. Kluwer Academic Publishers. Printed in The Netherlands.
[75] Sfard, A.: ???? Objectt….
[76] Sinko, M., & Lehetinen, E.: 1999, *The challenges of ICT in finnish education*. Helsinki, Finland: Sitra.
[77] Sutherland, R., Blacheff, N.: 1999, "Didactical complexity of computational environments for the learning of mathematics". *International Journal of Computers for Mathematical Learning* 4, pp. 1-26. Kluwer Academic Publishers. Printed in the Netherlands.
[78] The American Heritage® Dictionary of the English Language, Fourth Edition. Copyright © 2000 by Houghton Mifflin Company. Published by Houghton Mifflin Company. All rights reserved.
[79] Thomas, M., Tall, D.: 2001, "The Long-Term Cognitive Development of Symbolic Algebra". In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The future of the teaching and learning of algebra* (Proceedings of the 12th ICMI Study Conference), vol 2, pp. 590-597. Melbourne: The University of Melbourne. Australia.
[80] Thompson P.W., Thompson A.G. (1987): *Computer presentations of structure in algebra*. Proceedings of the Eleventh International Conference "Psychology of Mathematics Education PME – XI". Vol. I, pg. 248-254. Edited by J.C. Bergeron, N. Herscovics, C. Kieran. Montreal.
[81] Turkle, S., & Papert, S.: 1991, Epistemological Pluralism and the Revaluation of the Concrete. In I. Harel & S. Papert (Eds.), *Constructionism* (pp. 161-192). Norwood, New Jersey: Ablex.
[82] Valsiner, J., van der Veer, R.: 2000, "The Social Mind Construction of the Idea". Cambridge University Press.
[83] Vygotskij, L. S.: 1978, "Mind in Society. The Development of Higher Psychological Processes". Harvard University Press.
[84] Webster's Revised Unabridged Dictionary, © 1996, 1998 MICRA, Inc.
[85] Wilensky, U.: 1995, *Connected Mathematics – Building Concrete Relationships with Mathematical Knowledge*. PhD Thesis, Massachussets Institute of Technology.
[86] Yerushalmy, M., & Chazan, D. (2002). Flux in school algebra: Curricular change, graphing technology, and research on student learning and teacher knowledge. In L. D. English (Ed.), *Handbook of international research in mathematics education* (pp. 725-755). Mahwah, NJ: Lawrence Erlbaum.
[87] Yerushalmy, M., & Shternberg, B. (2001). Charting a visual course to the concept of function. In A. A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics*

(Yearbook of the National Council of Teachers of Mathematics, pp. 251-268). Reston, VA: NCTM.

[88] Zbiek, R. M., & Heid, M. K. (2001). Dynamic aspects of function representations. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The future of the teaching and learning of algebra* (Proceedings of the 12th ICMI Study Conference*,* vol. 2, pp. 682-689. Melbourne: The University of Melbourne. Australia.