Building theories: working in a microworld and writing the mathematical notebook

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In the framework of a long term teaching experiment we present an Educational approach based on the use of a dynamic geometry software and a symbolic manipulator. Here we present the general ideas of the followed approach focusing on how meanings can originate from phenomenological experience and evolve under the guidance of the teacher. In particular we will focus on meanings related to the ideas of theory, axiom and theorem.

INTRODUCTION

The research project, this paper will report on, started some years ago in the framework of a long term teaching experiment, which is to be considered a "research for innovation": action in the classroom is both a means and a result of the evolution of research analysis (Bartolini Bussi, 1996 p. 1). One of the main objective was to investigate the feasibility of a teaching approach centred on the use of the microworlds (Cabri-Géomètre and L'Algebrista), and aimed at developing theoretical thinking in both Geometry and Algebra (Mariotti, 2001, 200; Cerulli & Mariotti, 2002). Despite the differences between Algebra and Geometry teaching, a common educational approach was used, on which we are going to discuss: some aspects will be considered and some examples will be presented,

MICROWORLDS AND SEMIOTIC MEDIATION

The teaching experiment was carried a out and is still in progress at the 9th and 10th grade, level; it has been designed and developed within the vygotskian theoretical framework with particular reference to the *notion of semiotic mediation*. Given an artefact it can be used by the teacher to exploit communication strategies aimed at guiding the evolution of meanings within the class community ; this may be also the case of the computer which can be used by the teacher in order to direct the learner in the construction of meanings that are mathematically consistent (Mariotti 2002).

Our approach is based on the following general hypothesis: "*Meanings are rooted in the phenomenological experience (actions of the user and feedback of the environment, of which the artefact is a component), but their evolution is achieved by means of social construction in the classroom, under the guidance of the teacher*" (Mariotti 2002). Thus an artefact can be a source for the construction of *meanings* by its users, but consistency with Mathematics is not a priori guaranteed and needs to be built under the guidance of the teacher. As a consequence, activities within a microworld need to be interlaced with other social activities guided by the teacher in order to reach the construction of the mathematical meanings she is aiming to.

Based on these assumptions our approach is organised in the following cycle of activities:

1. Problem solving activities within and outside the microworlds: this is the field of phenomenological experience where we assumed meanings to be rooted.

2. Production of reports (written or oral) concerning problem solving activities: students' experience is fixed into signs on which collective discussion will be based.
3. Collective discussions, i.e. Mathematical Discussions according to the definition given by Bartolini Bussi (1998): starting from the produced reports the teacher tries to guide the class in the construction of socially shared meanings, consistent with didactical aims.
4. Production of reports concerning collective discussions: the results achieved in collective discussions become part of the class culture, and as such are expressed and fixed into written text, that may serve as a basis for future activities.

This cycle describes only the general structure of the teaching sequence and focuses on the main aspects we want to discuss in this paper. In particular the articulation between experiences centred on activities within the microworld and experiences centred on semiotic activities, consisting both in producing and interpreting texts.

WORKING IN A MICROWORLD AND WRITING A NOTEBOOK

The two microworlds share interesting features which according to the shared Vygotskian framework, are similarly exploited both the Geometry and the Algebra teaching experiments.

1. objects and commands can be thought as external signs of the fundamental elements of a corresponding mathematical theory (Geometry or Algebra).

For instance, basic *tools* are signs of *axioms* and *definitions of a Theory*; *new tools* may be introduced using a specific command (Macro construction in Cabri, Il Teorematore – i.e. Theorem Maker in L'algebrista); such new commands become signs of *theorems*;

2. actions within the microworld correspond to fundamental metatheoretical actions, concerning the construction of a theory.

For instance, *adding new buttons* to those already available corresponds to the meta-theoretical operation of *adding new theorems* to a theory. In the case of Cabri it is possible to create macros that synthesise geometrical constructions and that can be used at any moment. In the case of L'Algebrista it is possible to create new buttons representing equivalence relationships between algebraic expressions and that can be used at any moment by the user in order to transform an expression into another one.
Due to the described feature (for more details see Cerulli & Mariotti, 2002 Mariotti, 2001) L'Algebrista and Cabri result to be good potential environments for phenomenological experiences concerning the production and the use of theorems. Furthermore, they offer the possibility to experience the act of adding commands to the software. In other terms, once a semiotic link with mathematics is built (see Cerulli,2003), the two microworlds make it possible to directly experience the development of mathematical theories by proving and adding theorems, through the effective operations of creating and adding new commands.

Together with Cabri and L'algebrista, another specific tool characterises our experimentation: the *notebook* (ital. "quaderno di classe"). Each pupil is asked to edit a *notebook* where any result, discussed and socially accepted in the class, will be reported. In particular, it contains the list of the axioms and theorems (either in algebra or geometry) of the theory the class is working with, and when a new theorem is produced it is added to the list. Thus the notebook may be considered a representation of the culture and the history of the class, where the elements of the theory are fixed into ordered sequences, so that both the elements and their logic relationships are represented.

Collective discussions, edition of the *notebook* and writing reports, are different kinds of verbalisation activities. In the limits of this paper, we cannot carry out a detailed analysis of the dynamics between such different activities, in particular, taking into account the different registers (Duval,1995); in the following we may refer to all of them using the generic term "verbalisation activity" in order to distinguish them from “practical” activities taking place within Cabri and L'Algebrista. The reason why, within our approach, we use both activities in the microworlds and verbalisations is to be found in our basic theoretical hypotheses. In fact, within both the microworlds it is possible to realise phenomenological experiences concerning some aspects of mathematical activity, which are not so easily experienced in other environments. For instance, the fact that commands are signs of theorems and axioms makes it possible to use them as instruments. Such an instrumental approach to theorems gives proving activities a practical flavour, that it is impossible to be obtained otherwise (Cerulli 2002). In other terms, these two microworlds may offer a very rich phenomenological experience to pupils, in order to build specific mathematical meanings.

On the other hand microworlds put strong constrains (on purpose!) on the actions the user can perform, and thus also on the way he/she can express him/herself. Thus it seems important to have other environments with less constrains and more familiar to students. Furthermore, within Cabri and L'Algebrista, communication occurs between user and the machine, and is characterised by a rigid set of signs: the production of a new sign might then be inhibited. This could be an obstacle from a Vygotskian perspective, where the production of new signs is assumed to play a key role in the production and evolution of meanings, as it permits communication and involvement of new meanings into discourses.

For these reasons, we based our teaching experiment also on verbalisation activities. In fact, on the one hand they guarantee more expressiveness, on the other hand they facilitate the production of new signs to be used and shared in the social discourse, leading to production and evolution of meanings.

Once a practice is verbally expressed, it is possible to talk about it, and once the culture of the class is fixed in a notebook, it is possible to talk about it and eventually to compare it with what is written in the mathematics textbooks.

STRATEGY TO GUIDE THE EVOLUTION OF MEANINGS

According to our hypotheses, the meanings, arising from phenomenological experiences within microworlds, have to evolve, under the guidance of the teacher, towards the mathematical meanings the teaching/learning activity aims at. In our teaching experiments, the main structure of class activities can be schematised as in figure 1. Meanings originated in the phenomenological experience are shared within a collective discussion, fixed in the sets of command of Cabri and L'Algebrista and then reported in the personal *notebook*. Starting from this general idea we may consider the two different cases of axioms and theorems.

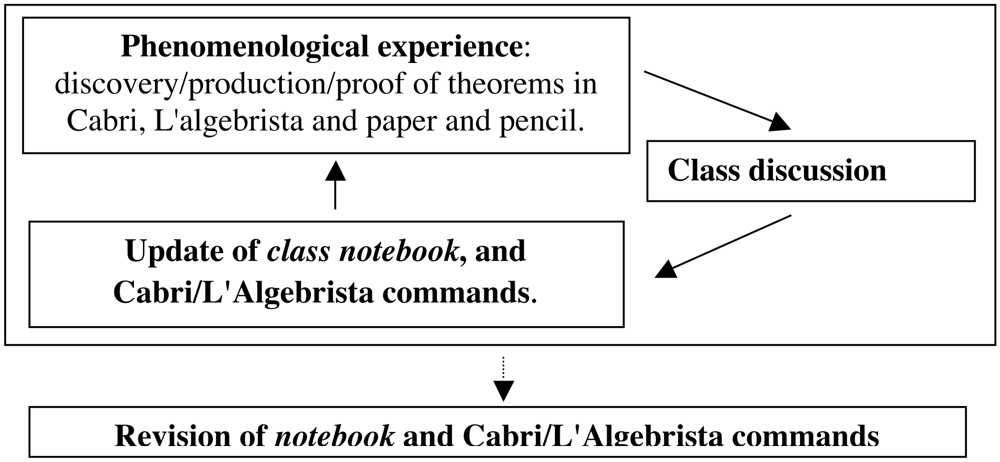

Figure 1 The main structure of the class activities

The case of axioms.

One possible way to introduce axioms is to begin working within a microworld: when it is firstly approached by the student, it presents a ready made set of commands. Such commands are *given* and can be used to work within the microworld. Thus the student *is faced with* a *given* set of commands that *are the only means of action* within the microworld, and that are actually *used to accomplish specific tasks*. Such an experience, under the guidance of the teacher, is then verbalised and socialised through a collective discussion, aiming at the formulation and the acceptance of a set of axioms, directly related to the given set of commands. Finally, each axiom is fixed into a statement on the *notebook*. Thus, at the end of this cycle, one obtains a set of commands in the microworld, a set of axioms in the culture of the class, and a set of statements in the notebook; furthermore, the fact that axioms are generated from commands, and statements from axioms, constitutes per se a link between them and may foster the idea that commands, and statements, are both signs representing axioms.

The case of a theorem.

Despite their differences, Cabri and L'Algebrista both allow the user to *create a new command*, *using given commands*. Starting from the new command and the sequence of actions producing it, a new *theorem*, with its *proof*, may be introduced in the culture of the class through verbalisation and socialisation, via collective discussion. The theorem is then fixed in the *notebook* as a statement together with a sequence of signs representing its proof.

Once introduced, theorems and axioms (and corresponding commands) can be used to accomplish new tasks, but their status in the culture of the class is different, as the processes generating them. Axioms originate from ready made commands, whilst theorems originate from command built on the commands already available. The dependence relationship, stated between new commands and the commands used to create them provides an operational referent to a logical structure in the organisation of the theory, as it is collectively built by the class.

When axioms and theorems are reported on the *notebook*, and new commands inserted in the microworlds, they may result to be ordered chronologically, however their different status, and the dependence relationships, may not always be evident. The activity of revising the notebooks gives the opportunity of reflecting and organising the set of axioms and theorems following their logical relationships.

The notebook (personal, but based on shared productions) together with the sets of commands of the microworlds, represent the "culture" and the history of the class. As a consequence, updating and revising them means to update the class culture. This is certainly a meta theoretical activity, corresponding to the construction of a mathematical theory; it may be interpreted as a phenomenological experience that raises meanings, that are then to be developed under the guidance of the teacher. The revision of the *notebook* (and updating of the corresponding set of commands) has to be interpreted from this point of view: it involves class discussions and writing of reports, and focuses on analysing the culture that has been produced along the history of the class.

In the following section we are going to discuss three example showing traces of the internalisation of the previous basic aspects, as a consequence of social activities.

THE STATUS OF AXIOMS AND THEOREMS: SOME EXAMPLES

The following examples are drawn form the data of our teaching experiments; the first concerns the proof provided by a pupil, the second an episode during a collective discussion and the last an excerpt from a report on a collective discussion.

Sum between monomials. After an activity within L'Algebrista and a collective discussion the theorem of the sum of monomials was introduced. Pupils are required to prove, in the paper and pencil environment, that "13*m+m*17=30*m". Elena, although not explicitly asked, gives two different proofs of the statement (fig. 2),. The first proof is produced using only axioms ("proprietà"), while the second one is produced using also the mentioned theorem, that she calls "Teorema 2". At each step the pupil indicates what axiom or theorem has been used to transform the expression: "com" stands for commutative property; "dist" stands for distributive property; "bottone di calcolo" stands for "button of computation", a command of L'Algebrista that executes only sums between numbers. Signs such as "bottone di calcolo" and the practice of underlying expressions show how symbolic manipulation, and proof of equivalencies between expressions are rooted in phenomenological experiences that take place in L'Algebrista (Mariotti &

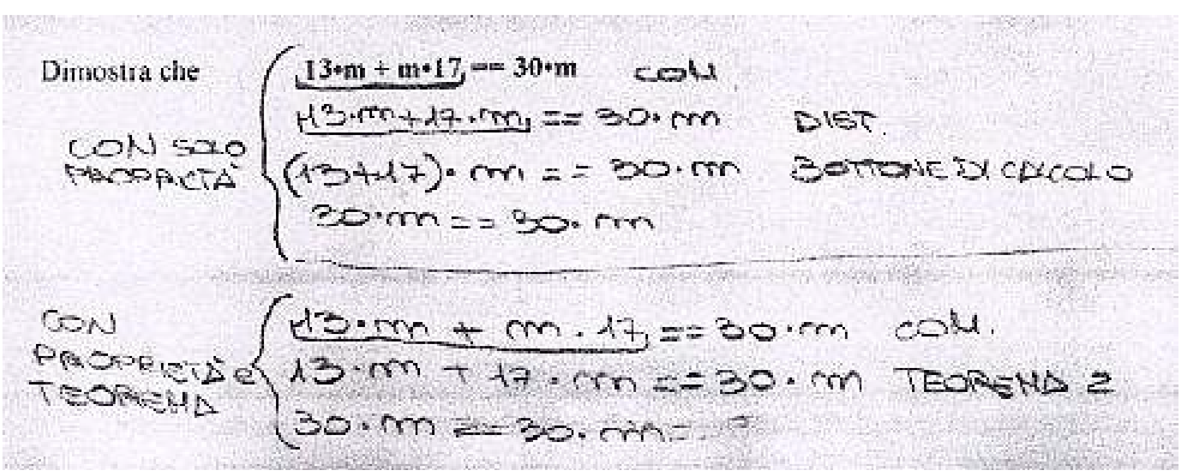

Figure 2 Elena's proofs

Cerulli, 2001). Finally, the fact that Elena writes "Teorema 2" originates in the social practice of the class of giving names to theorems and ordering them chronologically.

The Axiom/Theorem. In this episode the teacher (T) begins the lesson by asking pupils to recall what they said, 3 months earlier, about equations. At that time, they discussed the statement "**A=B ←→ A–B=0**", now she wants to start from this point in order to introduce the standard principles to solve equations.

Excerpt1

1. T: So, the first question is, do you remember what we have been doing at the end of last year? What did we focus on?
2. Tcl: The axiom theorem *(ita.: assioma teorema)*
3. Cri: axiom theorem one

[…]

6. T: What is it?
7. Tcl: if A is equivalent to B then A minus B is equivalent to zero.
8. T: come to write it *(on the blackboard)* and then explain why we called it axiom theorem

[…]

12. *Tcl writes on the blackboard:* a == b <=> a – b

[…]

14. T: do you remember why did we call it axiom theorem? Is it normal to call something "axiom theorem"?

[…]

20. Bzc: we didn't **know if…it was proved**, we took it as an axiom, last year, but **if later we are able to prove it** … we left it undecided.

In this activity, the history of the class becomes the source a new discussion. There is an element of the theory, which the class community decided to acquire and use even if its status it not clear, they "left it undecided" [20]; for the moment they take it as it is, but they know that in future they may go back to discuss its status: "but if later we are able to prove it…" [20].

The theorem of the bisector. After a first sequence of activities , the teacher sets up a collective discussion with the aim of revising the pupils' personal *notebooks*. From the comparison of the pupils' notebooks the teacher guides a mathematical discussion: the objective is that of ordering the sequence of the theoretical elements, as they are reported in the notebooks, and at giving them the right status: are they axioms, theorems or definitions? After the discussion each pupils is asked to write a report on such activity. During the discussion some time was devoted to the construction of angle bisector and the proof of the corresponding theorem called "*bisector theorem*" . In particular, different proofs were proposed, based on the application of different theorems. Trace of this part of the discussion can be found in the report of a pupil, and witnesses that the pupil Stefano writes:

We then switched to examine the proof of the bisector theorem, one of my classmates stated that the bisector theorem could be proved also with the isosceles triangle, but to do that we would have needed to have the last theorem concerning the perpendicular. If I say that, even having the theorem, we couldn't use it, ***it doesn't mean that we are fool*** *but simply that when we began [the proof]* ***we didn't have it, and our means for proving were in minor quantity****.*

CONCLUSIONS

A description of the general principles of our educational approach was given focusing on how some features of Cabri, L'Algebrista, and paper and pencil, may be used in order to foster the ideas of theorem, proof and theory. The phenomenological experience originated in the described environments may be exploited to produce mathematically consistent meanings; the revision of the history of the class may be used to build meanings related to the logical structures of mathematical theories.

The given examples show how pupils reached a good control in managing theoretical and meta theoretical aspects. Our basic hypothesis that this result is due to:

- the key role played by the class notebook as a store of the "class culture" and as input for class discussions
- the control of logical and chronological organisation of class findings obtained through the revision of the notebook

A better formulation, and a verification of such hypothesis is one of the focuses of our present research.

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