# A modelling challenge : untangling learner knowledge

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# A modelling challenge: untangling learners' knowing

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For now about three decades, systematic research on students learning of science and mathematics has evidenced a great variety of the possible understanding for a given notion, as well as an important sensitivity of these understandings to contexts. Students' knowing appears as a tangle of local conceptions of which we have a very partial and unorganised picture. We present here the outlines of the modelling tool we have developed, and of its framework, in order to give account of learners' knowings in a coherent way although preserving the realm of their diversity and heterogeneity.

## 1. FROM BEHAVIOUR TO MEANING

A key postulate of the theory of didactical situations, is that each problem-situation demands on the part of the student behaviours which are indications of knowing. This fundamental correspondence, established case by case, is justified by the interpretation of problem-situations in terms of game, and by the interpretation of behaviours in terms of indicators of strategies which adequacy must be demonstrated in the model—or representation—attributed to the student (Brousseau 1997, p.119). As Pichot wrote: "All behaviour implies a knowing" (1995, p.206).

Indeed, a behaviour depends on the cognitive characteristics of the person who performs it, as well as on the characteristics of her environment. But "person" and "environment" refer to complex realities of which not all aspects are relevant for the type of questioning we are considering. What is of interest for us is the person from the point of view of her relationship to a piece of knowledge. For this reason we will refer to the *subject* as a kind of projection of the person on her cognitive dimension. In the same way the environment does not interest us in all its complexity, but only by its features which are relevant with respect to a given piece of knowledge. We will call *milieu* such a subset of the subject' environment. The milieu is a kind of projection of the environment on its cognitive dimension. But, being interested in learning complex knowledge, we must extend the classical idea of *milieu* in order to integrate symbolic systems and social interaction as means for the production of knowings. Let us take the case of mathematics: in this domain knowings are not the results only of the interaction between a subject and a material milieu, but they involve also interactions with systems of signifiers produced by the subject herself, or by others.

The meaning of milieu we adopt here, is in fact the one Brousseau (1997 p.61) proposed: *the milieu is the subject's antagonist system* in the learning process. Hence, we do not consider knowing as a property, which can be ascribed only to the subject, nor only to the milieu. On the contrary, we consider it as a property of the interaction between the subject and the milieu, the subject's antagonist system. This interaction is meaningful because it succeeds in fulfilling the necessary conditions for the viability of the *subject/milieu system*. By viability we mean that the subject/milieu system has a capacity to recover an equilibrium following some perturbation. In some cases, the nature of this equilibrium may even change if these perturbations are such that this is

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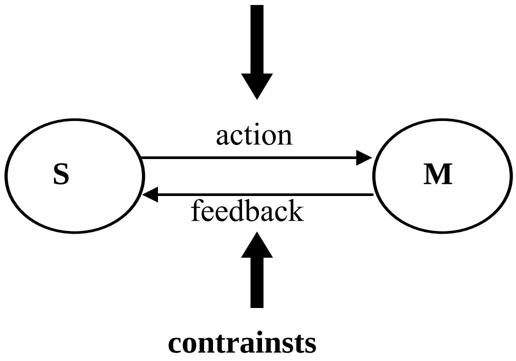
necessary. This is, in other words, a formulation of Vergnaud's postulate that problems are sources and criteria of knowings (1981 p.220). Problem, in our model, means a more or less severe perturbation of the subject/milieu system. The existence of a knowing can then be witnessed by its manifestation as a problem-solving tool, what is reified as behaviours of the subject/milieu system while it overcomes the perturbations in order to satisfy its constraints of viability. These constraints do not address the way the equilibrium should be recovered but the criteria this equilibrium should satisfy (we could also say that there is not only one way to know.) Following Stewart (1994 pp. 25-26) we would say that these constraints are proscriptive.

We can now propose a definition of the meaning of a piece of knowledge, that is "a knowing", which can be pragmatically and efficiently used in a didactical *problématique.*

> *A knowing is characterised as the state of dynamical equilibrium of an action/feedback loop between a subject and a milieu under proscriptive constraints of viability.*




In a didactical *problématique* we are interested in the nature of the proscriptive constraints that the subject/milieu system must satisfy, since it may be by a reproduction of these constraints that a given learning might be obtained. But learning in a didactical setting (school, training centre, etc.) has to satisfy other constraints than those one in general experiences in "natural" settings. Among these constraints, not yet known exhaustively, we can mention two which are specific to didactical settings: time constraints and epistemological constraints (Arsac and al. 1992). The former is due to the way educational or training settings are organised (duration of the school life, organisation of the school year, organisation of the lessons, etc.). The latter is due to the existence of a knowledge of reference which underlies any content to be taught and which *de facto* provides criteria for the acceptability of a learning outcome.

The role of the teacher, with respect to a given content to be taught, is to organise the encounter between a subject and a milieu so that, from their interaction, a knowing—which can be considered as acceptable with respect to the didactical intention—can emerge. Such an encounter is not a trivial event. To be in an environment is not enough for the student to be able to "read" in it the milieu relevant with respect to the teaching purpose. To select the relevant features of the environment, to identify its feedback and to understand it with respect to the intended target of the action is not self-evident. The means for the teacher to succeed in supporting students learning is to construct a situation which allows the devolution to them of both the milieu and the relevant relationships (action/feedback) to this milieu. But the more the teacher supports and stimulates students, the more he or she may be unable to obtain the intended learning outcomes since students may then behave to please the teacher and not to solve the problem as expected. This is a basic complexity of didactical systems.

Learning, in the approach we present, is a process of reconstruction of an equilibrium of the subject/milieu system, which has been lost following a perturbation of the milieu, or of the constraints on the system. The didactical *problématique* considers the case of perturbations provoked deliberately, with the intention to provoke learning. One should realise that there is a perturbation only if there is a gap recognised by the subject between the expected result of her action and the actual feedback from the milieu. This means on one hand that the subject should be able to recognise the existence of a gap not acceptable with reference to her intention, and on the other hand that the milieu can provide identifiable feedback (with a pocket calculator any convergent sequence is stationary).

Sometimes no gap is recognised by the subject whereas we—as observers—identify that one should have been recognised. We call this unnoticed gap an *error* only when it is the symptom of a knowing, that is: the symptom of the robustness of a previous equilibrium of the subject/milieu system (as opposed to a mere slip). For the construction of certain knowings it is necessary to be aware of some errors and to overcome them. For example, uniform convergence is a very powerful concept whose construction often proceeds via errors due to a common idea of conservation of certain properties of functions (like the statement, false in general, that the limit of a series of continuous function is a continuous function).

The knowing source of errors could be locally refuted, but could keep some validity even with respect to the knowledge of reference which may allow to express its domain of validity (some students could see uniform convergence as a restriction of simple convergence). But even when this erroneous knowing is rejected and replaced by a new knowing, it may keep a pragmatic validity (decimal numbers are not natural numbers with a dot, but to consider them as such is quite useful insofar as computation is concerned).

The consequence is the possibility of ancient knowings persisting despite their refutation, simultaneously with new knowings and giving the idea of an incoherent and unstable subject. Here is the starting point of the construction of the tangle of human knowings.

## 2. THE TANGLE OF LEARNER KNOWING

### 2.1 COHERENCE AND SPHERE OF PRACTICE

Some researchers have evidenced the paradox of the co-existence of a rational thinking together with knowings which, from the observer's point of view, seems to be contradictory (e.g. Bourdieu 1980). Basic explanations of this phenomenon, are time arguments on one hand, and the diversity of situations on the other hand. Time organises the subject's actions sequentially in such a way that contradictory knowings are equally operational because they appear at different moment of her history: the contradictory knowings can then ignore the one the other. The diversity of the situations introduces an element of a different type, it is a possible explanation insofar as one recognises that each knowing is not of a general nature but that on the contrary it is related to a specific and concrete domain of validity on which it is acknowledged as an efficient tool.

Following Bourdieu, we will refer to *sphere of practice* in order to designate these domains of validity mutually exclusive in the history of the subject. In a sphere of practice the rational subject is reconciled with the knowing subject. We must insist on the fact that the contradictions which are evidenced in this way, are recognised as such by an observer who is able to relate situations which are seen as independent and completely different by the subject herself.

In the *observer referential system*, these observed states of the subject/milieu system should nevertheless be labelled in the same way. Just as we usually speak of the subject knowing of decimal numbers, of continuity of functions, of energy, of electricity or of digestion even if later on one would complain that this knowing is not coherent. But indeed, to accept the existence of contradictory knowings seems to refute the theoretical principle of a mental construct which is a product of a process of adaptation ruled by criteria of reliability, robustness and adequacy to problem-solving or task-performing. To overcome this difficulty we first re-introduce the notion of conception—a heritage from research in education—we will then propose a more formal characterisation in order to proceed in our modelling process.

### 2.2 FROM MISCONCEPTIONS TO CONCEPTIONS AS KNOWINGS

Research on learners' misconceptions has developed during the 80s, essentially reporting on what one called naive theory, private concepts, beliefs, or even of the mathematics or the physics of the child (Confrey 1990). In all this work, the child-learner is seen as a subject fundamentally different from the adult-expert who appears as the owner of the knowledge of reference (ibid. p.29). This view did not exclude the recognition of some sort of cognitive legitimacy of misconceptions:

> […] *a child may not be "seeing" the same set of events as a teacher, researcher or expert. It suggests that many times a child's response is labelled erroneous too quickly and that if one were to imagine how the child was making sense of the situation, then one would find the errors to be reasoned and supportable*. (ibid. p.29)

In other words: a misconception has a domain of validity; if not, it would not exist! So there is a very short distance between a misconception and a knowing. The key difference, for researchers of that time, is that for any misconception it exists a refutation, which is known at least to the observer. Even later, when *misconception* gained the status of a knowing, it remained the idea of the existence of an *intrinsically correct* knowledge of reference—a position that is clearly refuted by our knowledge of the history of science and mathematics.

Bachelard (1938, p.13) wrote in a nice manner that reality is never what one could believe, but is always what one should have thought of [1]. Knowledge is always in progress. If we accept this, errors witness the inertia of the instrumental power of

[1] "Le réel n'est jamais " ce qu'on pourrait croire " mais il est toujours ce qu'on aurait dû penser" (Bachelard 1938, p.13).

knowledge, which has proved its efficiency in enough situations. One may notice that constructivism did not immediately recognise this nature of error, or at least less than one might think. Even from the Piagetian point of view, the learner is seen as a cognitive subject but not yet as a fully knowing subject: *from the functional level, the child is identical to the adult, but with a mental structure which varies depending on the stages of development* (Piaget 1969, p.224). Engaged in a construction process the child *is always obliged to accommodate herself to an external reality, to the peculiarities of the environment from which she has to learn everything* (ibid. p.225). The content of this mental structure of the child has not yet completely the status of a knowing, even though all theoretical ingredients exist to allow to consider it as such. The Copernican revolution was not achieved at the beginning of the 70's.

The main evolution is to recognise that *errors are not only the effect of ignorance, of uncertainty, of chance […], but the effect of a previous piece of knowledge which was interesting and successful, but which now is revealed as false or simply not adapted.* (Brousseau 1997, p.82). The thesis of Brousseau at the beginning of the seventies goes beyond the fact of recognising as knowings the mental constructs source of errors. It states that some of these knowings likely to be falsified are necessary to learning. Even more: the trajectory of the student may have to pass by the (provisional) construction of erroneous knowings because the awareness of the reasons why this knowing is erroneous could be necessary to the understanding of the new knowing. Following Bachelard, Brousseau calls *epistemological obstacles* these compulsory gateways to a new understanding.

The main difference between the previous position, inherited from the misconception paradigm and the one we support here, lies in their epistemological meaning. The former implies the existence of a knowing of reference intrinsically true (decontextualised); the later requires only to establish a relationship between two knowings with the idea of an evolution without judgement on each of them. Any knowing is what it is, whether it appears to be erroneous or not, partial or badly adapted. A knowing is first of all the result of an optimal adaptation of the subject/milieu system following criteria of adequation and efficiency.

## 3. CONCEPTION, A THEORETICAL MODEL IN MATHEMATICS

The word *conception* has been used for years in research on teaching and learning mathematics and science, but it has been used as a common sense idea rather than explicitly defined. It functions as a tool but its definition remains implicit; it is not taken as an object of study as such (Artigue 1991, p.266). This very informal way to develop research on learning and learners conceptions in education results in a severe difficulty in establishing a meaningful relation between various research results, as well as an impossibility to develop good relationships with people involved in the design and implementation of computer-based learning environments. So, we develop a model to provide an answer to both issues: unifying the modelling of learners' conceptions, providing a tool, which could be utilised, for the design of learning environments. This model is based on a formalisation able to give account of the multiplicity of the conceptions referring to the same notion, and of the possible contradictions in a subject's knowing. From this definition of conception, we will derive definitions of the term knowing and concepts as abstract entities whose differences lie in their functions and relations.

The definition we give of *conception* is first formal, which means that its adequacy to a pragmatic or a common sense understanding of what is a conception should elaborated a posteriori. Second it is limited to mathematics, as an essay, not meaning that this approach cannot be extended, but that its extension has a price: it should be performed under the epistemic vigilance of the other disciplines—a price we could not afford until now.

Hence, we call conception C a quadruplet (P, R, L, ∑) in which:

- P is a set of problems;
- R is a set of operators;
- L is a representation system;
- ∑ is a control structure.

The semantic of P is that it is the sphere of practice of the conception. The issue of its actual characterisation is complex, and even open in most cases. One option would be to consider all the problems for which a given conception provides efficient tools to elaborate a solution. This is the option suggested by Vergnaud in the case of additive structures (1991 p.145)[2]. Another option could consist of considering a finite set of problems with the idea that other problems will derive from them. This is the solution proposed by Brousseau (1997, p.30). The first option is not specific enough and fails to help; the second option leaves open the question of establishing that such a generative set of problems can be constructed for any conception.

The issue of the characterisation of the set of rules and of the representation system is more classical. Any attempt at designing a computational model of knowledge offers a possible solution to it; it is at the core of research in artificial intelligence—especially in the case of learning environment. The last dimension of a conception, the control structure is often left implicit, although one may realise that the criteria which allow to decide whether an action is relevant or not, or that a problem is solved is a crucial dimension of a mathematical conception.

It is important to insist on the fact that this characterisation of a conception does not relate more to the subject than to the milieu with which he or she interacts. On the contrary, it allows a characterisation of the subject/milieu system: the representation system allows the formulation and the use of the operators by the active sender (the subject) as well as the reactive receiver (the milieu). The control structure allows expressing the means of the subject to decide of the adequacy and validity of an action, as well as the criteria of the milieu for selecting a feedback. This characterisation provides us with a cognitive tool to analyse and model learning, without depending on a mental model in a psychological sense.

## 4. AN EXAMPLE, FUNCTIONS

The theme of function has been studied extensively. It is classical nowadays to consider a priori the following categories:

- Function as a correspondence "law";
- Function as symbolic expressions;
- Function as a graphical object.

Actually, these categories correspond to the three main semiotic registers associated to "function" (numerical, algebrical, graphical), or the three classical periods in history (Kepler, Euler, Newton).

Vinner (1992) identifies eight components of students' conceptions of function, among which the followings reproduced here as examples:

[2] This definition proposed by Vergnaud was in fact coined at the beginning of the 80's, (cf. Vergnaud 1981).

*A function should be given by one rule;*
*The graph of a function should be regular and systematic;*
*A function is a one-to-one correspondence.*

Since this seminal work of Vinner, started in 1983, these features of students understanding of function have been confirmed all over the world. But as they are presented, these features are not yet organised into conceptions. In particular one misses indications about their domain of validity as well as of the way they could be implemented in a problem-solving situation.

The idea that a graph or a curve, from the students' point of view, should exist in relation to an algebraic expression is central, both having to conform to certain constraints (like being drawable). Graph and curve are two different entities, which must be distinguished: curve refers to a geometrical object—which may be represented by an equation, graph refers to a representation of a function in the graphical register (one plots a graph). This distinction underlies two types of student conceptions, the Curve-Algebraic and the Algebraic-Graph. Let us describe them using the tools just presented:

- *Curve-Algebraic* conception: $C_{CA}=(P_{CA}, R_{CA},$ *Graphic-symbolic*, $\sum_{CA})$,
- *Algebraic-Graph* conception: $C_{AC}=(P_{AC}, R_{AC},$ *Symbolic-graphic*, $\sum_{AC})$

These two conceptions have in common the same semiotic registers, both the symbolic and the graphical, but with different degrees of importance in both cases. In the case of the curve-algebraic conception, the criterion is that the curve must have an algebraic representation. In the case of the algebraic-graph conception the criterion is that the algebraic representation must be associated with a graph which one should be able to draw. But the difference between both conceptions, to the eye of an observer, is not straightforward; it is by looking at the rules-tools they require in problem-situations and at their control structure that we can shape the distinction (Gaudin 1999).

Its is often referred to the history of mathematics with the idea of finding there some support to understanding students' difficulties in learning calculus. We must emphasise here that students' spheres of practice are radically different from those of the mathematicians that historians consider. The didactical system has first introduced students to "good" functions, mainly playing with two different settings: algebraic and graphical. As Sierpinska (1989 p.17) noticed, shapes of graphs of elementary functions can be prototypes of conceptions.

Artigue (1992 p.130) gives an excellent example of this phenomena when she analyses, in the case of the qualitative approach to differential equations, the *false theorem*: "if f(x) has a finite limit when x tends toward infinity, its derivative f'(x) tends to 0". She wrote:

> *When sketching solution curves, we draw the simplest one compatible with the identified set of constraints, but in doing this, we add extra constraints concerning the convexity that can be expressed roughly in the following way: convexity has to be the least changing possible or, in algebraic terms, the sign of f'', for a solution f, has to be the most constant possible. So, f' is implicitly the most monotonic possible.* (ibid. p.130)

She concludes that the mentioned false theorem can be seen as an instantiation of the hypothesis: *if for x large enough, f' is monotonic, f' has necessarily a limit (finite or not) and the unique limit compatible with a horizontal asymptote is 0. In other words, by adding the condition f' monotonic, this false theorem becomes a true one* (ibid.) Such an analysis of a tool (which is the actual status of theorems in students' practice) used by students evidences the interaction of the graphic and the algebraic register and the role played by the characteristics of the sphere of practice.

Depending on the curriculum they have been exposed to, students have available more or less sophisticated tools to analyse some elementary algebraic formulas and to describe the behaviour of the corresponding functions. These spheres of practice may be described in detail following a close analysis of textbooks, which are available to students.

## 5. CONCEPTION, KNOWING AND CONCEPT

We will not give a detailed account of the benefit we get from the characterisation of conception outlined here (see Balacheff 1995, 1995a). We will shortly conclude by evoking how, in this framework, we can relate the notions of conception, knowing and concept, which are at the core of any learning research.

The relation between knowing and conception is rather straightforward: a *knowing* is a set of *conceptions*. This allows us to speak of the domain of validity of a knowing—the union of the domain of validity of the related conceptions—but at the same time we can acknowledge the contradictory character of a knowing (one conception is false from the point of view of another -- cf., p.233). Let us now call *concept* the set of knowings sharing the same content of reference (for a characterisation, see Balacheff 1995, in particular the way to show that two knowings share the same object).

Then, a conception is the instantiation of the knowing of a subject by a situation (it characterises the *subject/milieu system in a situation*), or it could be considered as the instantiation of a concept by the pair (subject/situation). From the relationships between conceptions induced by the definition adopted here, and from their properties, we can draw in a natural way properties and relationships between knowings, as well as between concepts.

## RÉFÉRENCES


Arsac G., Balacheff N., Mante M. (1992) Teacher's role and reproducibility of didactical situations. *Educational Studies in Mathematics* 23 (5) 5-29

Artigue M. (1991) Épistémologie et didactique. *Recherches en didactique des mathématiques.* 10(2/3) 241-285.

Artigue M. (1992) Functions from an algebraic and graphic point of view: cognitive difficulties and teaching practices. In: Dubinsky E., Harel G. (eds.) *The concept of Function*. (MAA Notes Vol. 25, 109-132). Mathematical Association of America.

Bachelard G. (1938) *La formation de l'esprit scientifique*. Paris : Vrin.

Balacheff N. (1995) Conception, connaissance et concept. In : Grenier D. (ed.) *Didactique et technologies cognitives en mathématiques, séminaires 1994-1995* (pp.219-244). Grenoble : Université Joseph Fourier.

Balacheff N. (1995a) Conception, propriété du système sujet/milieu. In : Noirfalise R., Perrin-Glorian M.-J. (eds.) *Actes de la VII° Ecole d'été de didactique des mathématiques* (pp.215-229). Clermont-Ferrand : IREM de Clermont-Ferrand.

Bourdieu P. (1980) *Le sens pratique*. Paris : Les éditions de Minuit.

Brousseau G. (1997) *Theory of didactical situations in mathematics*. Dordrecht: Kluwer Academic Publishers

Confrey J. (1990) "Misconceptions" accross subject matters: charting the course from a constructivist perspective. In: Proceedings of the *Annual meeting of the American Educational Research Association*.

Gaudin N. (1999) *Caractérisation des conceptions du concept de fonction*. Mémoire de DEA. Grenoble: Laboratoire Leibniz.

Piaget J. (1969) *Psychologie et pédagogie*. Paris : Éditions Denoël.

Pichot A. (1994) Pour une approche naturaliste de la connaissance. *Lekton* 4(2) 199-241.

Sierpinska A. (1989) *On 15-17 years old students' conceptions of functions, iteration of functions and attractive fixed points*. Institut de Mathématiques, preprint 454. Varsovie: Académie des Sciences de Pologne.

Stewart J. (1994) un système cognitif sans neurones : les capacités d'adaptation, d'apprentissage et de mémoire du système immunitaire. *Intellectika* 18, 15-43.

Vergnaud G. (1981) Quelques orientations théoriques et méthodologiques des recherches françaises en didactique des mathématiques. *Recherches en didactique des mathématiques*. 2(2) 215-231.

Vergnaud G. (1991) La théorie des champs conceptuels. *Recherches en didactique des mathématiques*. 10(2/3) 133-169.

Vinner S. (1992) The function concept as a prototype for problems in mathematics education. In: Dubinsky E., Harel G. (eds.) *The concept of Function*. (MAA Notes Vol. 25, 195-213). Mathematical Association of America.