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MEANINGS FOR FRACTION AS NUMBER-MEASURE BY EXPLORING THE NUMBER LINE

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Construction of meanings for fraction as number-measure is studied during the implementation of exploratory tasks concerning comparison and ordering of fractions as well as operations with fractions. 12-year-old students were working collaboratively in groups of two with software that combines graphical and symbolic notation of fractions represented as points on the number line. Fractions as points and segments, ordering fractions as part of kinesthetic activities and abstracting the scaling of the numerical unit on the number line are some of the meanings developed.

THEORETICAL BACKGROUND

In this paper we report research [1] aiming to explore the mathematical meanings constructed collaboratively by 12 year-old students concerning the notion of fraction as a number-measure depicted on the number line within a specially designed computational environment. The students worked in collaborative groups of two using the ‘Fractions Microworld’ (FM) [2], a piece of software which combines graphical and symbolic notation of fractions represented as points on the number line. The research perspective and task design adopt a constructionist approach to learning (Harel & Papert, 1991), focusing particularly on students’ interaction with joint representations (Kynigos, 2002) to construct mathematical meaning.

In the research literature understanding fractions involves the coordination of many different but interconnected ideas and interpretations such as part-whole, measure, quotient, operator and ratio (Lamon, 1999). The interpretation of a fraction within part-whole relations is the first and probably the most dominant facet of the concept presented to students at the primary level, after which the algorithms for symbolic operations are introduced. Whereas algorithmic competencies in the domain of fractions are usually fairly developed, they are mainly associated with a mechanistic use of fractions in calculations while understanding is usually weaker as well as the competencies to solve problems including fractions especially in number-measure situations (Aksu, 1997). These situations of fractions are usually accompanied by the pictorial representation of a number line and students are expected to measure distances from one point to another by partitioning certain distances from zero in terms of some unit. The value of the resulting fraction in this case comprises *the number* (i.e. the rational number that the fraction represents), while the distance on the number line *the measure*. This dual reference of partitioning to quotients as well as to distances from zero can be seen as an example of the complexity of situations in which the number-measure interpretation of fractional numbers occurs.

The existing research results confirm that the measure interpretation of fractions on the number line is one of the most difficult for the students to acquire since it is related to the well documented hidden discontinuities between natural and fractional numbers (Stafylidoy & Vosniadou, 2004) such as the uniqueness in the symbolic representation of natural numbers, which does not hold for fractions (i.e. several fractions can represent the same fractional number), or the density of fractional numbers depicted on the density of number line (i.e. between any two fractions there is an infinite number of fractions). As children's experience with numbers is usually based on the discrete integers used for counting, their theory of number may become increasingly resistant with age to accepting fractions as numbers that represent the continuous nature of measurable points in space. This situation is aggravated by the apparent difficulty of students to relate the number line with their real-world knowledge or some kind of external realistic grounding (Marshall, 1993). Meaningful examples that incorporate pupils' previous experiences may be difficult to derive especially at the younger ages. In the absence of a "mental model" for filling the gaps between the integers, children are likely to overgeneralize their use of counting numbers in a way which provides a formidable obstacle to accommodating their theory of number to accept fractions (Siegal & Smith, 1997). Another constraint concerns the students' confusion over the nature of unit on the number line. As fraction is usually perceived by pupils through the part-whole metaphor, the unit is considered as bigger than all fractions and many times the integer number line is treated as a unit rather than the segment from zero to one (Baturo & Cooper, 1999).

Although there are certain difficulties in conceptualizing the number line representation of fractions there is a recent resurgence of interest in the representational potential of number lines for the learning of fractions (Ni, 2001, Hannula, 2003, Charalambous & Pitta-Pantazi, 2005). Entailing "a dynamic movement among an infinite number of stopping-off places" (Lamon, 1999, p. 120), working with fractions on the number line has been considered as critical for pupils to conceptualise the different subconstructs necessary for the deep understanding of the concept of fraction in general (Hannula, 2003). Recently, the representational infrastructure of computer-based environments provided us with tools to reconsider the role of the number line in making the number-measure facet of fractions more accessible and meaningful to children. In this study experimentation with fractions is considered as more flexible and dynamic for the students due to the available tools that combine graphical symbolic notation allowing manipulation of the provided representations. Under this perspective we emphasised not on closed 'didactical goals' but on pupil's active construction of meanings as they operationalised the use of the available tools while making judgments, taking decisions and developing situated abstractions (Noss & Hoyles, 1996) during the problem solving process. The concept of fraction as measure was not considered on its own but in terms of the concepts tightly related to it, the situations in which it it may be used and the available representations (i.e. within a conceptual field, Vergnaud, 1991).

TECHNOLOGY AND TASKS

In our research perspective we attribute emphasis on two aspects of the pedagogical setting that are likely to foster mathematical learning: the computer environment and the task design. The construction of a fraction in the FM is realised as a quotient of a division: the divider and divisor are selected from one horizontal and one slanted line respectively (see Figure 1). After the selection of two numbers from the slanted lines it is also provided instantly a geometrical representation of a fraction based on the Thales Theorem. Since this theorem is introduced at the secondary level, we chose to bypass this interpretation in the present study and to use it as a 'black-box' for the students. The horizontal line is called 'number line' while the slanted one is called 'multiplication (or partition) number line'.

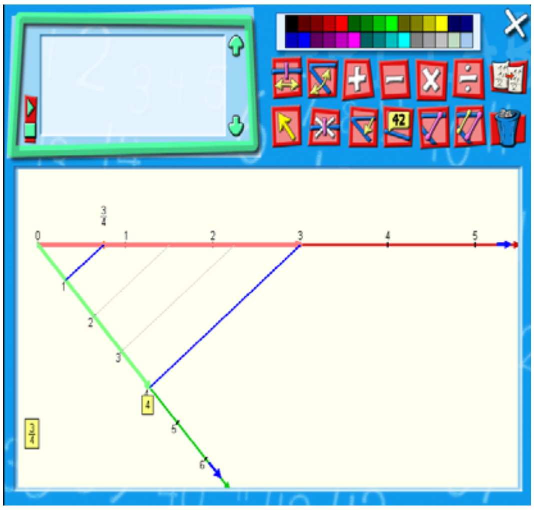

Figure 1: ¾ represented on the FM horizontal number line.

The symbolic notation of each fraction is automatically given near its representing point on the horizontal line in a post-it form. In the FM there is thus a representation of fraction as a number (resulting by the division between the numerator and the denominator) as well as a measure (represented by a point on the number-line). Other features of the microworld, like the ability to modify dynamically the size of the numerical unit (i.e. the distance between two successive integers) in both half lines and to perform basic arithmetic operations on fractions (i.e. addition, subtraction and multiplication) are parts of the visual imagery and manipulative aspects of the tool. The novel character of the above representations can thus lead to the identification of a 'distance' between the mathematical objects constituting the representation of fractions in the microworld (tool design) and those found in the traditional curriculum, based primarily on the part-whole scheme. For example, the symbolic representation of fractions with a numerator equal to 1 coincides with the part-whole representation of these fractions (e.g. the position of the fraction 1/3 indicates also the respective part-whole relationship, 1 part of the 3 in which the unit 1 is divided) which does not happen with any other type of fractions as represented in the FM. Moreover, the lack of any observable partitions of the numeric unit on the horizontal line can be considered as a characteristic that enhances the abstract nature of the representation of fractions as points among integers.

In task design it was adopted a perspective in which the 'distance' between tool design and aspects of didactic knowledge of fractions was considered as a challenge to design exploratory activities which may provoke multiple pupil's responses concerning comparison and ordering of fractions as well as operations with fractions.

This decision was also reinforced by the fact that the FM only represents fractions in specific ways and does not signal some kinds of mistakes by means of visual feedback. This characteristic of the tool gives space for pupil's interpretations of the given feedback which can be seen as a point of negotiation among the students rather than as a closed answer. The above choices led to the design of two strands of problems (see Tables 1 and 2) based on integrating the representation of fractions on the number line with places in an everyday context. In pupils activity thus the mathematical nature of fractions was planned to integrate with their use to measure quantities in authentic problem situations in which the move in different points on the number line was connected with the idea of persons covering certain distances.

(1) George's house is 1 kilometer far from his school. On his way to school he sees a square at 1/2 km, his friend's Chroni's house at 1/3 km and a sweetshop at 1/6 km. Can you say in which order he sees them when coming back home?
(2) (a) Constantina's school is 1 km away from her house. On her way to school she sees a kiosk at 6/7 km, a super market at 2/5 km and a playground at ¾ km. Which is the order she sees them on her way to school?
(b) Lazaros is Constantina's best friend; his house is between the playground and the super market. Can you find some fractions indicating the position of his house?

Table 1: The first strand of activities.

(3) Efi and Constantina are friends. They meet each other at the playground. Efi says to Constantina: "You are very lucky. Your house is closer to the playground than mine." Discuss about the position of Efi's house.
(4) Constantina says to Efi: "I think that you are lucky too. You walk only 2/3 kilometers to go to school." Why is Efi lucky? Can you find the exact position of Efi's house? What is the distance of the two friends' houses?
(5) Maria, a friend of Constantina and Efi, says: "I believe that the luckier of you is the one who walks less in going both to the school and the playground every afternoon." Who do you believe is luckier?

Table 2: The second strand of activities.

METHOD

The experiment was carried out as a case study at the computer laboratory of a primary school in Athens with four 6th grade (last grade of the primary level in Greece) students divided in two groups (G1 and G2) consisted of one boy and one girl. These pupils had already been introduced to the notion of fraction in the traditional classroom. Each pair of students was assigned to one computer. Four 90-minute meetings were conducted by each group of pupils. A team of two researchers participated in each data collection session using one camera and two tape-recorders. All that was said in each group was captured by one tape-recorder. One researcher was occasionally moving the camera to both groups to capture the overall activity and other significant details in pupils’ work as they occurred. The researchers occasionally intervened to ask the students to elaborate on their thinking with no intention of guiding them towards some activity or solution. Verbatim transcriptions of all audio-recordings were made. For the analysis, we adopted a generative stance (Goetz & LeCompte, 1984) allowing for the data to shape the structure of the results and the clarification of the research issues. The identified illustrative episodes can be defined as moments in time which have particular and characteristic bearing on the pupil’s interaction with the available tools accompanied with the constructed mathematical meanings.

RESULTS

Fractions as points and segments on the number line

In their efforts to conceptualize certain aspects of the tasks, pupils have constructed their own pictorial representations of the number line as a means to visualize the contextual information they associated with the notion of fraction on the number line (Problem 1). We observed that this kind of activity appeared only at the beginning of the experiment, highlighting the students’ attempts to link their intuitions about fractions, the representation of numbers provided by the tool and the conceptualisation of the given tasks.

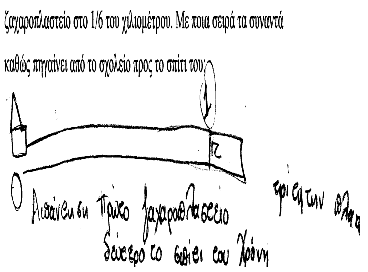

Figure 2: Individual representation of the number line (Problem 1a).

As far as the individual representations became part of pupils’ activity, their interactions with the computer environment became strongly associated with the respective points on the screen. Pupils started to speak of them as if they were points, though, from a mathematical point of view, are measures of distances. According to the language of the problem at hand, students continued to refer to the numbers on the FM number line either as points or segments. For instance, when they were calculating the position of certain places they were showing points, while when they were calculating certain distances, they were showing parts of the number line.

Kinesthetic and conceptual aspects of ordering fractions

Another aspect of the student’s conceptualisation of fractions concerned the emergence of body-syntonic activities in ordering fractions by value. The use of the tool’s functionalities at that time was interwoven not only with the available representational infrastructure of the tool but also with the task design. This was achieved by students’ interpretation of the feedback concerning the position of fractions on the number line as feedback concerning the representation of a distance in an everyday context. This particular idea, in turn, facilitated the integration of the provided representation of fractions with pupils ‘walking’ on the number line to reach the places mentioned in the given problems.

S1: [*Showing with her figure the distances on the number line of the screen*] So, it is twice this distance. She went from home to school, came back, went to playground and then came back home again.

S2: [*Trying to confirm*] She went to school, came back and from home … to playground.

In these lines, students (Group 1) display an attempt at making sense of the total distance covered by Constantina as described in Problem 3. As their utterances reveal, the conceptual realm remains subjected to the kinesthetic experience by the metaphor that the child “walks along” the number line indicating points (places) and distances (segments). In the same group this kind of activity lead to the elaboration of ordering fractions based on the comparison of the directed position of certain points from zero (i.e. from the left to the right) in terms of the unit distance on the number line. In trying to find some fraction for the position of Lazaros’ house, situated between the super market and the playground (Problem 2b), pupils were engaged in experimenting with the tool to find out some fractions between 2/5 and ¾. The following excerpt highlights students’ artifact-mediated activity regarding the role of the denominator in the value of fractions.

R: How did you manage to find out this fraction [*i.e. 3/5*]?

S1: We played with the denominators. For instance, the fraction ¾. We chose a bigger

denominator, which decreases the fraction, and we estimated that it will fall in between.

R: Did you increase the numerator also?

S1: Yes. Let’s try one.

There is a dynamic aspect in ‘playing’ with the values of denominator and numerator as it is expressed here by S1. The plurality in keeping up the partitioning process seems to play a leading role in conceptualising the order of fractions as well as the nature of the ongoing partitioning itself. Based on the coordination between their existing knowledge (i.e. the value of a fraction decreases when the denominator increases) and the computer feedback, both groups of students explored the continuity of the number line and constructed meanings related to the notion of infinity of fractions that exist between two fractional numbers. In the following excerpt, for instance, students (Group 2) had already constructed twelve fractions between 2/5 and ¾ following the above method and appear ready to continue.

R: So, there are surely 12 fractions Do you think that you could find more?

S3, S4 [*Together*]: Yes.

S3: There could be made many. We can fill all this, from here up to there [*Shows in the computer screen the segment in the number line between 2/5 and ¾*], with fractions.

In this context pupils used the tool to verify their conjectures on the exact position of certain fractions on the line or to check the results of their calculations in paper and pencil. However, due to specific functionalities of the tool both groups had difficulties in connecting the results of certain operations with the solution to Problem 5 which needed the addition of integral and fractional parts of the number line. This was mainly related to the fact that the symbolic representation of a calculation in the FM includes only the numbers of the respective operation (e.g. 1/3 + 1/2) and not the resulting equivalent fraction (e.g. 5/6). Moreover, the user is not able to measure a specific part of the number line (e.g. between ½ and 3/4) if it is not starts from zero. So, the measure of a specific segment on the number line is realized by its equivalent transformation starting from zero. Both groups of students did not succeed in coordinating the conceptual part of the artifact-mediated operations that recognised as necessary for the solution with the representation of the respective fractional amounts on the number line. Considering Constantina's house on zero, for instance, students (Group 1) found that Efi's house is on the point 5/3, while the playground is on the point ¾. In solving Problem 5, S1 needed to compare ¾ with the result of the subtraction 5/3-3/4.

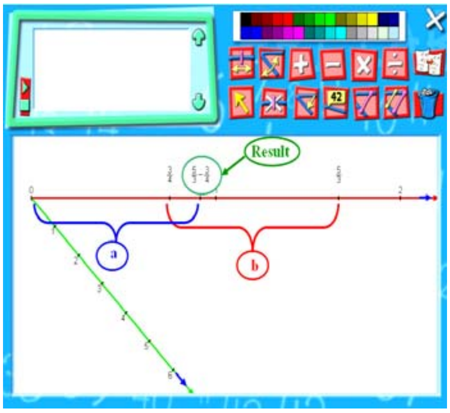

Figure 3: Two interpretations of the subtraction 5/3 – ¾.

R: How did you find that point [*shows the point labeled as 5/3-3/4 on the number line*]?

S1: I subtracted 5/3–3/4. Because 5/3 is the whole and I subtracted the part from zero up to the playground. So, we 've finished.

R: Ok. Can you answer now who walks longer?

After calculating the result of the above subtraction (11/12) in the paper, S1 answers:

S1: Eleven twelfths.

Though completed the requested calculations concerning the distance 'b' (Figure 3) and transformed it to the (equal) segment starting from zero (i.e. the distance 'a' represented by the expression 5/3-3/4 on the screen), S1 is unable to connect it with the paper and pencil difference consisted of one fractional number (11/12). The difficulty in distinguishing the equality of the above expressions is obviously related to the limitation of the educational exploitation of the representations and functionalities of the FM in the present study.

Abstracting the position and the size of the numerical unit

One of the most critical conceptual jumps that contribute to the children's difficulty in learning fractions concerns the role of the unit. In number-measure situations particularly, this problem seems to arise from the fact that children are unable to decide what constitutes an appropriate 'unit' on the number line, since may be induced to taking any whole line segment shown as representing a unit, rather than the line segment between two successive integers. The unit is thus at the core of the main mechanisms that play a major role in naming distances form zero as well as in understanding the relation between the size of the unit and the location of specific points on the number line. In our experiment students' conceptualization of unit appeared in different phases of their explorations as an indication of their progressive familiarisation with the control of the mathematical nature concerning the construction and representation of fractions in the FM. In several cases distinguishing that the numerical unit of measure was regardless of its position on the number line was facilitated by the preceding kinesthetic interpretation of the proposed tasks. In the next excerpt students (Group 2) integrated the (unit) distance between Constantina's house (corresponding to 0) and the school (corresponding to 1) in their approach and used it to bypass the constrain of representing numbers on the left of the zero point of the FM number line. After experimenting with different positions of Efi's house in relation to Constantina's house and to school (Problem 3), S3 tries to explain to S4 the possibility to consider Efi's house on the left of the point 0. In doing so, he also indicates that the distances between different places (i.e. points) are independent from the position of the unit on the number line.

S4: Where is Constantina's house? We know that it is 1 kilometer far from school.

S3: At 0.

S4: How do we know that?

S3: We symbolised it here (*He shows the distance from 0 to 1*). We can do the same with 3 and 4 or 9 and 10. We just preferred 0 and 1 for school. Do you understand?

It is noticeable in the above excerpt that the indexical gestures of S3 on the number line appear as part of the situated abstraction concerning the position of the unit, to indicate in a precise way its imagined different positions on the number line. In a similar way, students abstracted also the independency of the length of the numerical unit from the fractional parts of it since, once determined, it remains so throughout a problem. This was mainly achieved through the extensive use of the scaling functionality of the FM by which students could dynamically change the measure of the numerical unit.

R: Why do you enlarge it?

S3: Because they were stuck together and they seemed as of they were 67.

R: Yes, they are more discernible, but are they the same? Does anything change?

S3: No, just to discern them.

Triggered by the need to zoom on specific parts of the number line, S3 enlarges the numerical unit indicating a suitable understanding of the fractions-as-measures used to determine distances on the number line. This also implies the conceptualisation of ratio scales which require both equal intervals and an absolute zero.

CONCLUSIONS

Dickson et al. (1984) pointed out that many children are likely to be uncertain about the nature of a fraction as a number-measure represented by a point on the number line well into the secondary age range. In this study however, some interesting meanings around the measure personality of fractions seemed to have emerged while 12-year-olds were using the symbolic, graphical and manipulation tools of the FM s/w. Working within these tools, the dual representation of fractions as numbers and points on the number line, the order of fractions and the role of the unit came into play offering us an interesting terrain in which to investigate the nature of pupil's engagement in meaningful experimentation with the fractional amounts of their activities to measure distances on the number line and compare them in mathematically efficient ways. The foregoing episodes illustrate the pupil's progressive focusing on connections between the measure interpretation of fractions to the other concepts, situations and representations of the relative conceptual field. In the first part of the analysis an icon-driven individual representation of the number line seemed to mediate pupils' familiarisation with the representations provided by the tool. The second part of the analysis showed clearer how the proposed tasks provided a context for the students to coordinate the interplay between the points as static numbers to the distances as dynamic measures from zero. In these cases, the mathematisation of pupils' responses in ordering fractions was inextricably related to the kinesthetic nature of the computer feedback translated in the context of the given activities. Students' inability to conceptualise the equality of fractional expressions and its representations provided by the tool signalled the development of an interesting domain to extend the educational potential of the software in a future extension of the present study. In the last part of the results pupils' previous experience with the FM tools had been moving in the direction of scaling the numerical unit on the number line by abstracting both its position and its size. It is thus suggested that the dynamic access to static representations provided by digital media, like the FM, could possibly (a) change our conceptions about what can be learnable at school, in what way and at which grade and (b) indicate a need to reconceptualise the existing curricula with the integration of technology.

NOTES

1. The research took place in the frame of TELMA (Technology Enhanced Learning in Mathematics), a European Research Team (ERT) established to focus on the improvements and changes that technology can bring to teaching and learning activities in mathematics. TELMA is part of Kaleidoscope, a Network of Excellence funded by the European Community (IST-507838).

2. The FM is part of AriLab2, a stand-alone version of an open system developed by the Consiglio Nazionale delle Ricerche - Instituto Tecnologie Didattiche (CNR-ITD) research team in Genoa (Italy). AriLab2 is composed of several interconnected microworlds based on the idea of integrated multiple representations and functionalities designed to support activities in arithmetic problem solving and in the introduction to algebra.

REFERENCES

Aksu, M. (1997). Student performance in dealing with fractions, *Journal of Educational Research* 90(6), 375-380.

Baturo, A. & Cooper, T. (1999). Fractions, reunitisation and the number-line, *Proceedings of the 23rd PME Conference*, Haifa, Israel, Vol. 2, 81-88.

Charalambous, C. & Pitta-Pantazi, D. (2005). (in greek) The role of number line in the evaluation of the notion of fraction as number-measure, *C. Kynigos (ed.) Proceedings of the First Conference of the Greek Association of Research in Mathematics Education*, University of Athens, Greece, 404-414.

Dickson, L., Brown, M. & Gibson, O. (1984). *Children Learning Mathematics: A Teacher's Guide to Recent Research*, London, Cassel Educational Ltd.

Goetz, J. P. & LeCompte, M. D. (1984). *Ethnography and Qualitative Design in Educational Research*, Academic Press, London.

Hannula, M.S. (2003). Locating fraction on a number line, *Proceedings of the 2003 Joint Meeting of the PME and PMENA*, University of Hawaii, Vol. 3, 17-24.

Harel, I. and Papert, S. (1991). *Constructionism*, US: Ablex Publishing Corporation.

Kynigos C. (2002). Generating cultures for mathematical microworld development in a multi-organisational context, *Journal of Educational Computing Research*, Baywood Publishing Co. Inc. (1 and 2), 183-209.

Lamon, S. J. (1999). *Teaching Fractions and Ratios for Understanding*, NJ: Lawrence Erlbaum Associates.

Marshall, S.P. (1993). Assessment of Rational Number Understanding: A Schema-Based Approach, in T. P. Carpenter, E. Fennema, T. A. Romberg (Eds.), *Rational Numbers: An Integration of Research*, Lawrence Erlbaum Associates, 261-288.

Ni, Y. (2001). Semantic domains of rational numbers and the acquisition of fraction equivalence, *Contemporary Educational Psychology*, 26, 400-417.

Noss, R. and Hoyles, C. (1996). *Windows on Mathematical Meanings*, Kluwer.

Siegal, M., & Smith, J.A. (1997). Toward making representation count in children's conceptions of fractions, *Contemporary Educational Psychology, 22,* 1-22.

Stafylidou, S., & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fraction, *Learning and Instruction*, 14(5), 503-518.

Vergnaud, G. (1991). La théorie des champes conceptuels, *Researcher en Didactique des Mathématique*, 10(2/3), 133-169.