Cabri-géomètre as a mediator in the process of transition to proofs in open geometric situations.

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CABRI-GÉOMÈTRE AS A MEDIATOR IN THE PROCESS OF TRANSITION TO PROOFS IN OPEN GEOMETRIC SITUATIONS:

An exploratory study

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Abstract

In this paper I will present some findings of a teaching experiment on proof. Pupils were requested to produce conjectures in open geometric situations, to validate and, finally, to prove them. These activities took place in the microworld Cabri-Géomètre Our project was aimed to investigate the validity of the Cabri software as a mediator in the process discovering-conjecturing-proving. We found out that different dragging modalities are crucial for producing a shift from conjecturing to proving: these modalities can be analysed as the perceptive counterpart of the cognitive processes students use.

INTRODUCTION

A crucial point of current discussion among researchers in mathematics and mathematics education is the concept of proof (for example see Balacheff, 1988; Duval, 1992; Barbin, 1988; Chazan, 1993; Moore, 1994; Thurston, 1995; Boero et al, 1996; Hanna, 1996; Harel & Sowder, 1996; Simon, 1996; Mariotti et al, 1997).

One characteristic of the debate concerns the role of explorations and conjectures with respect to proofs. In the literature, there is a wide range of opinions concerning this issue: some authors have underlined the central role of formal proofs in mathematics, in opposition to the heuristics used in the discovery process, while some others have stressed the fundamental role of heuristics compared to formal proofs. Moreover, many papers stress the fact that in the process of solving a problem, a crucial point is a dialectic between the exploratory phase and the subsequent phase in which all the bits discovered informally are reorganised into validated statements. There are different opinions with respect to this dialectic: for example the proof schemes classified by Harel & Sowder (1996), Simon's "transformational reasoning" (1996), "proofs that explain" by Hanna (1996), the "cognitive unity" of Mariotti et al. (1997) and Polya (1957) underline a kind of continuity in the dialectic. On the contrary other authors, as Duval (1991), underline the cognitive and epistemological gap between argumentation and proof. In an intermediate position we find

Balacheff (1988) analysis of the phases of transition to proof (naïf empirism, generic example, formal proof).

This paper, drawing on ongoing Italian research (Boero et al, 1996; Mariotti et al, 1997; Arzarello et al, 1998a; 1998b), stresses the importance of the construction of proofs, pointing out not only their formal aspects of established products but especially the fact that, as processes, they are deeply rooted in the activity of producing conjectures as a whole. According to this perspective, research should aim at the development of suitable learning environments, which can support students in the transition from explorations and conjectures to more formal hypothetical reasoning and proofs. Classroom experiments provide evidence for the fact that cognitive continuity can be 'constructed' on the basis of the production of conjectures; that is, students engaged in activities which require explorations of a situation and production of conjectures, are more likely to be able to organise a proof at the end, than if presented with an established statement and asked to prove it.

The problem becomes even more interesting when new technologies, such as Cabri-Géomètre, Geometer's Sketchpad, Excel, Derive, are used in the classroom as tools for exploring, conjecturing, validating and even proving theorems.

In this paper I take into consideration the example of Cabri-Géomètre (Laborde & Laborde, 1992). The possibility of dragging things around the screen is one of the most important affordances of Cabri. Having a set of movable points give students new material to observe, which they do not have in paper and pencil. The kind of geometry you can do with this type of software is referred to as dynamic geometry. However this term could possibly have a broader meaning, not only related to the experimentation rendered possible by this software. Hölzl (1996) talks about a *Cabri-geometry*. Even if at first sight the drag-mode is not a new construction tool, and so it should not alter Euclidean geometry, implementing dragging requires the users to start from different assumption and to perform different operations on the objects. "Line segments that stretch and points that move relative to each other are not trivially the same objects that one treats in the familiar synthetic geometry, and this suggests new styles of reasoning" (Goldenberg, 1995).

A lot of research is currently being carried out about the potentialities of dragging, all of them underlying many different aspects. Laborde (1995) stresses the importance of the dragging function in order to understand the relationship between the notions of 'drawing' and 'figure'. The 'dragging test' provides an introduction to the theoretical meaning of a geometrical construction, moving away from the only practical problem (Mariotti, 1996).

Dragging supports the production of conjectures: exploring figures by moving them allows users to discover invariant properties of geometric figures (Hanna, 1989). The possibility of dragging offers a feedback to the discovering phase (Laborde, 1995), and in this way it provides support to the role of proofs as real "explanations" of a conjecture or property (Hanna, 1989).

Drawing on this background, our research group developed a teaching experiment on proof, which we carried out last year in Italy. The project involved a classroom activity addressed to the second year of high school (pupils aged 15-16). Students were requested to explore open geometric situations (Arsac at al, 1988). These tasks differ from the traditional problems “prove that…”, because they involve questions such as “what happens to the figure if…, which different configurations can you find…”, so that pupils are required to produce conjectures, to validate them and finally to prove them. These problem-solving activities took place in the microworld Cabri-Géomètre.

The project was aimed at:

- investigating whether a cognitive continuity can be supported in the mental processes that make students shift from conjectures to proofs;
- testing the validity of the software Cabri as a mediator (Vygotskij, 1992) that can be a bridge between the activity of discovering and formulating conjectures and that of proving them.

This paper gives an account of some findings related to this project[1] and provides issues for further discussion.

In section 1. I briefly describe a theoretical model, based on both theoretical considerations and empirical observations, which describes the way conjectures are produced by experts and how they manage the transition from the conjecturing to the proving phase. In section 2. and 3. I describe the different dragging modalities we observed analysing the students’ processes while solving the geometric problems and I present an example to illustrate them. In section 4. the dragging modalities are analysed from the point of view of the theoretical model. Finally I present a fine analysis of a protocol of a pair of students, which is a case in point for our analysis.

[1] The complete analysis of the project is in Olivero (1998)

1. A THEORETICAL MODEL

Prior to the project in school, we analysed the performances of experts (mathematics teachers in high school and at University) dealing with elementary but non trivial open geometric problems in paper and pencil. They were asked to think aloud while solving the problems.

This analysis ended up in a theoretical model which describes the way conjectures are produced by experts and how they manage the transition from the conjecturing to the proving phase (Arzarello at al, 1998a; 1999).

The main points of the model are the followings.

- **Ascending control** (Saada-Robert, 1989; Gallo, 1994).

This is the modality according to which the solver 'reads' the figure in order to make conjectures. The stream of thought goes from the figure to the theory, in that the solver tries and finds the bits of theory related to the situation he is confronted with. This modality relates to explorations of the given situation.

- **Abduction** (Peirce, 1960; Magnani, 1997).

In the model, abduction means choosing 'which rule this is the case of', that is the subject browses his theoretical knowledge in order to find the piece of theory that suits this particular situation. Explorations are transformed into conjectures.

According to Peirce, of the three logic operations, namely deduction, induction, abduction (or hypothesis), the last is the only one "which introduces any new idea; induction does nothing but determine a value, and deduction merely evolves the necessary consequences of a pure hypothesis. Deduction proves that something must be; induction shows that something actually is operative; abduction merely suggests that something may be." (CP, 5.171). Abduction looks at facts and look for a theory to explain them, but it can only say a "might be", because it has a probabilistic nature. The general form of an abduction is:

> a fact A is observed
>
> if C was true, then A would certainly be true
>
> So, it is reasonable to assume C is true.

An example illustrates this concept. Suppose I know that a certain bag is full of white beans. Consider the following sentences: A) these beans are white; B) the beans in that bag are white; C) these beans are from that bag. A deduction is a concatenation of the form: B and C, hence A; an induction would be: A and C, hence B; an abduction is: A and B, hence C (Peirce called hypothesis the abduction). (Peirce, 1960, p.372).

- **Descending control** (Gallo, 1994).

This modality occurs when a conjecture has already been produced and the subject seeks for a validation. He refers to the theory in order to justify what he has previously 'read' in the figure and validates his conjectures.

The model shows that abduction plays an essential role in the process of transition from ascending to descending control, that is from exploring-conjecturing to proving. Abduction guides the transition, in that it is the moment in which the conjectures produced are written in a logical from 'if…then'; all the ingredients necessary for the proof are already present. This model suggests an essential continuity in the process exploring-conjecturing-validating-proving, for experts.

2. DRAGGING MODALITIES IN CABRI

We formulated the hypothesis that a cognitive continuity can be stimulated also in novices' performances, provided suitable mediators and environments are supplied. We used the cognitive model as an analytical tool to analyse students' strategies exploited in exploring geometric situations in Cabri: in this microworld abduction has a special role, that makes explicit the role of the Cabri environment as a mediator which favours the transition process.
First of all, observing[2] how students use the mouse while solving a problem in Cabri, we determined different dragging modalities.

- **Wandering dragging:** moving the basic points on the screen randomly, without a plan, in order to discover interesting configurations or regularities in the figures.
- **Bound dragging:** moving a semi-dragable[3] point (it is already linked to an object).
- **Guided dragging:** dragging the basic points of a figure in order to give it a particular shape.
- ***Lieu muet* dragging**: moving a basic point so that the figure keeps a discovered property; that means you are following a hidden path (*lieu muet*), even without being aware of this.
- **Line dragging:** drawing new points on the ones that keep the regularity of the figure.
- **Linked dragging:** linking a point to an object and moving it onto that object.
- **Dragging test:** moving dragable or semi-dragable points in order to see whether the figure keeps the initial properties. If so, then the figure passes the test; if not, then the figure was not constructed according to the geometric properties you wanted it to have.

3. A GENETICAL DEVELOPMENT OF THE DRAGGING MODALITIES

Let us see how these dragging modalities can be successfully exploited in an open problem of exploration, thus leading the solver to a range of discoveries.

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[2] An observer in the classroom (the author) observed a pair of students each session, paying attention both to what they said and to what was happening on the screen.
[3] A semi-dragable point is a point on an object, that can be moved but only on the object it belongs to.

TASK[4]:

You are given a triangle ABC. Consider a point P on AB and the two triangles APC and PCB. Make an hypothesis about the properties of ABC which are necessary so that both APC and PCB are isosceles (such triangles are called 'separable').

We can find two different configurations:

fig.1: AP=PC=CB

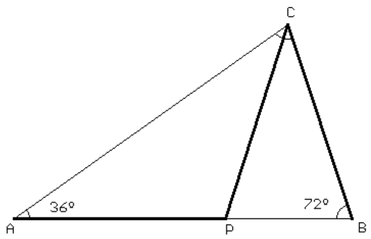
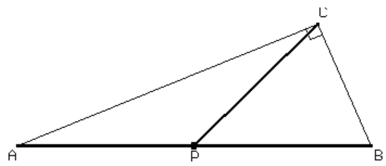


fig.2: AP=PC=PB



We investigate the configuration of fig.2

What are the characteristics of the triangle in fig.2?

In Cabri you draw a triangle ABC and P as the midpoint of AB. Then you start moving the point C, which is a 'dragable point' (Hölzl, 1996), all around the screen in order to see whether such triangles (the 'separable' ones) exist. You are doing **wandering dragging.** Through this way of dragging, you find many triangles that satisfy the property (fig. 3), so now you are sure that the task has a solution.

Now you go on moving C and you stop when the triangle ABC is 'separable', such as when PC equals AP. Looking at the figure you start thinking about the characteristics of this triangle. In order to know more, you try to drag C in such a way that ABC keeps its property (PC=AP). In other words, your dragging is no more 'by chance': you feel that you are moving along a main direction, the one which allows the triangle ABC being divided into two isosceles triangles. Therefore you are using ***lieu muet*** **dragging**. While in Cabri I the path followed by C cannot be seen, so it is muet, in Cabri II you could visualise it[5] and perceive that it resembles something known: it is similar to a circle (fig. 4).

To understand the situation better you can exploit the **line dragging** by marking the points correspondent to the positions occupied by C when ABC has the right property (fig. 5). Now you observe that they seem to lie on a circle, exactly on the circle centred

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[4] This problem is also discussed in Hölzl (1995; 1996).

[5] By means of the menu command 'Trace on/off': it traces the path of a selected object as it moves. It is not present in Cabri I.

in P with radius PA (=PB). The *lieu muet* now becomes explicit; by constructing the circle with centre P and radius PA you see that the points previously marked really are on that circle (fig. 6). So you conjecture that the 'separable' triangles are those whose vertex C belongs to a circle centred in P and with radius PA. This condition is equivalent to the fact that ABC is inscribed in a circle with AB as a diameter, which means that the angle <ACB is right. Therefore your conjecture can now be more precisely formulated in a logical form: if ABC is right-angled (<ACB=90°) then it is 'separable'.

In Cabri you have got a way to validate this statement. By linking the vertex C to the circle discovered by means of line dragging and moving it on that, so using **linked dragging**, you can see that all the triangles that are continuously redrawn satisfy the property of being 'separable' (fig. 7).

As a last step you can construct a right-angled triangle ABC, the middle point P of AB, the segment PC and use the **dragging test**, such as move the triangle through all its dragable points and observe that it keeps the asked property (fig.8).

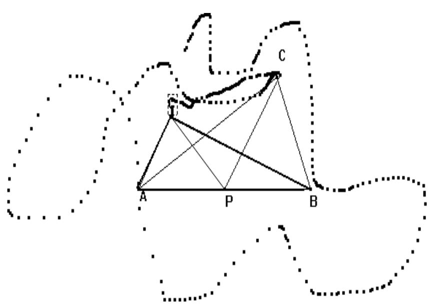


Fig.3

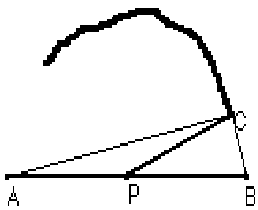


Fig.4

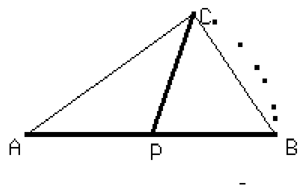
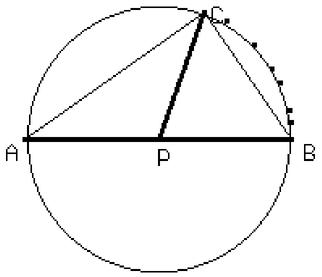
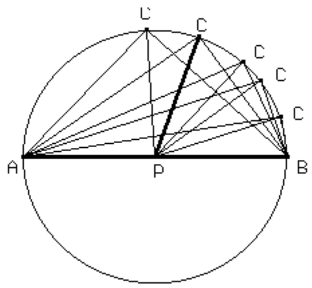


Fig.5





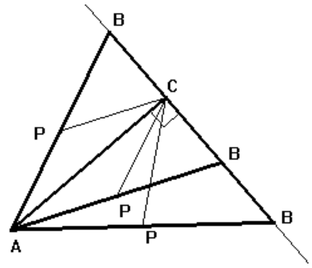


Fig.6 Fig.7 Fig.8

The first level of analysis, just from a perceptive point of view, shows that there is a 'genetic' hierarchy in the use of these dragging modalities, in that the solution process develops through a sequence of different modalities. This 'genesis' is not prescriptive, in that not all solutors will undertake it.

4. DRAGGING AND THE THEORETICAL MODEL

The next step concerned the analysis of these modalities from the point of view of the theoretical model. Dragging in Cabri seems to show at a perceptive level what the students' cognitive processes are. Actually, we realised that they exploit these different dragging modalities in order to achieve different aims.

- **Wandering, bound and guided dragging** are used to investigate and explore a given task, so they are part of the ascending control stream.
- ***Lieu muet* dragging** can be seen as a wandering dragging which has found its path; the trace of this dragging represents, at an empirical level, a locus that is not yet visible to the subject. A *lieu muet* can act both as a producer of new powerful heuristics (Holzl, 1996) and as a logical reorganiser of the previous investigations (Pea, 1987). This modality reveals the beginning of the shift from ascending control to descending control, that is an abduction. The solver is beginning to see a certain relation/property/invariance and he is trying to make sense of it in logical terms[6]. Therefore lieu meut dragging supports students in producing abductions, and as a consequence in the transition between the two modalities of control.
- **Line dragging** follows *lieu muet*, as it makes the locus explicit and visible on the screen; it is part of the process of transition towards descending control.

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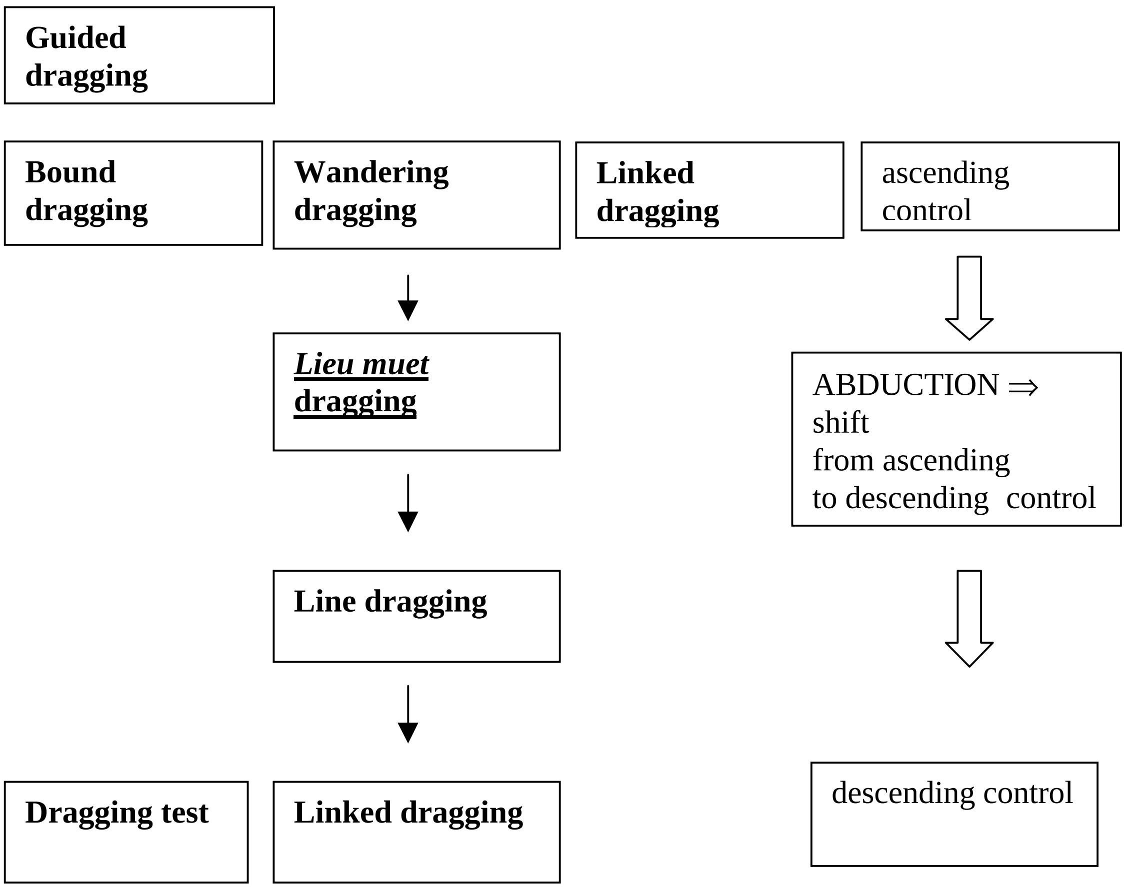
[6] If drawn figures keep some regularity, then the point C describes a certain locus L. If C runs on L, then the corresponding figures F(C) show some regularity, invariance or rule.

- **Linked dragging** allows the subject to check his conjecture: if the locus can be constructed in Cabri (e.g. a line, a circle), the subject can link the point to that locus, and the discovered property must be kept through dragging it onto the locus. Therefore it reveals the beginning of descending control. On the other hand it can also be exploited in the exploration process, for example if you want to decrease the free parameters of the initial situation.
- **Dragging test** is used as a means of validating a conjecture, in particular conjectures that are originated by a visual or a construction, therefore it reveals a kind of descending control.

At the end of this analysis, it is clear that the transition from one dragging modality to another shows a 'genesis', as represented in the following table.



5. A CASE IN POINT IN STUDENTS' PRODUCTIONS.

I now illustrate an example.

> TASK: Let ABCD be a quadrilateral. Consider the bisectors of its internal angles and their intersection points H, K, L, M of pairwise consecutive bisectors. Drag

ABCD, considering all its different configurations: what happens to the quadrilateral HKLM? What kind of figure does it become?

This problem was given to a class of 27 students (15 years old), who were asked to solve it working in pairs at the computer. After one hour a classroom discussion took place in order to discuss the conjectures and proofs produced by the students. An observer (the author) took notes of the activity of two pairs of students.

We analyse the solving process of one pair of students.

WHAT STUDENTS DID	DRAGGING MODALITIES and THE COGNITIVE MODEL
Episode 1	
At first students start exploring the situation by examining standard cases. It seems they are following an implicit rule: a) When ABCD is a parallelogram, HKML is a rectangle b) When ABCD is a rectangle, HKLM is a square. c) When ABCD is a square, HKLM is a point (fig.1).	They use **guided dragging** in order to get different shapes of ABCD. **Ascending control** is guiding their experiments, as their aim is to get some conjectures about the configuration. The last step allows them to see a degenerate case: HKLM disappears into one point.
Episode 2	
As soon as they see that HKLM becomes a point when ABCD is a square, they consider it an interesting fact, therefore they drag ABCD (from a square) so that H, K, L, M keep on being coincident. They realise that this kind of configuration can be seen also with quadrilaterals that apparently have not any common property.	Now a regularity is discovered; so they use ***lieu muet*** **dragging**. They drag ABCD so to keep the property they have just found out. They are still in the stream of **ascending control**, as they are exploring the situation, but now they have a plan in their mind: they look for some common properties to all those figures which make HKLM one point.
Episode 3	
Paying attention to the measures of the sides of the figure ABCD (which appear	Even if the locus is not explicitly recognised by the students, it is this kind of dragging that

automatically next to the sides and change in real time, while dragging), they see that the sum of two opposite sides equals the sum of the other two (fig.3); they remember that this property characterises the quadrilaterals that can be circumscribed to a circle.	allows them to discover some regularity of the figures. Here they make an **abduction**, because they select 'which rule it is the case of': this is the case of circumscribed quadrilaterals. Referring to the example by Peirce, we can say that: A is "the sum of two opposite sides equals the sum of the other two", B is "a quadrilateral is circumscribed to a circle if and only if the sum of two opposite sides equals the sum of the other two", i.e. something you know while C is "these quadrilateral are circumscribed". Their reasoning is: A & B, then C. Once they have selected the right geometric property, they can 'conclude' that this is the case of circumscribed quadrilaterals. The conditional form is virtually present: its ingredients are all alive, but their relationships are still reversed, with respect to the conditional form; the direction after which the subjects see things is still in the stream of the exploration through dragging, the control of the meaning is ascending, namely they are looking at what they have explored in the previous episodes in an abductive way.
Moreover, using the Cabri menu, they construct the perpendicular lines from the point of intersection of the angle bisectors to the sides of ABCD: they see that this point has the same distance from each side of ABCD, then they draw the circle which has this length as radius: it is the circle inscribed in ABCD. After that they formulate a conjecture: *If the external quadrilateral can be*	The direction of control now changes: here students use the construction modality (and the consequent dragging test) to check the hypothesis formulated through abduction and at the end they write down a sentence in which the way of looking at figures has been reversed. By *lieu muet* dragging, they have seen that when the intersection points are kept to coincide the quadrilateral is always circumscribed to a circle. Now they formulate the conjecture in a logical

circumscribed to a circle, then its internal angle bisectors will all meet in one point, so the distances from this point are equal and the sum of the opposite sides is equal too.	way, which reverses the stream of thought: if the quadrilateral is circumscribed then the points coincide.
Episode 4	
At the end they construct a circle, a quadrilateral circumscribed to this circle, its angle bisectors and they observe that all of them meet in the same point (fig.4).	At the end they check their conjecture. Now they are using the dragging test and their actions show **descending control**.

The process that takes place can be summarised as follows:

a) First, they see (in Cabri) that:

If H, K, L ,M are coincident (one point) then the sum of two opposite sides equals the sum of the other two.

b) Then they make an ABDUCTION, that is 'quadrilaterals that can be circumscribed to a circle'.

c) Finally, they produce a conjecture in a logical form:

If ABCD circumscribed then H, K, L, M coincide (one point).

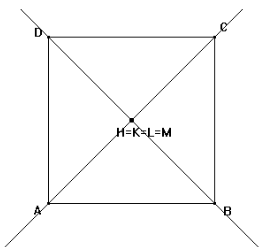


fig.1

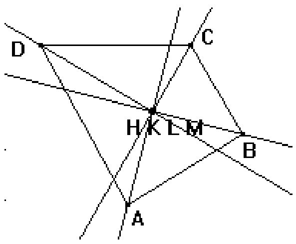


fig.2

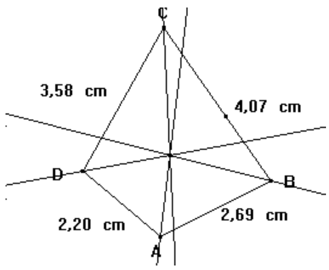


fig.3

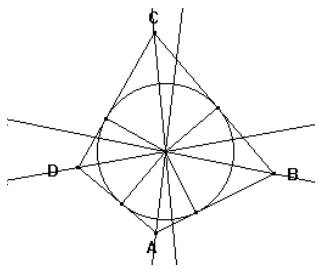


fig.4

FINAL REMARKS AND OPEN QUESTIONS

Summarising, the main results we produced are the following.

- The use of dragging in Cabri changes with respect to the control students have of the situation.
- There is a genetic development of dragging modalities in Cabri: this allowed us to define a hierarchy of dragging modalities, which is the perceptive counterpart of the cognitive development during a solution process. The genesis we showed is not something

prescriptive, in that not all solution processes will undertake it. However the overall important point is that this shows that dragging acts as a mediator. The way the transition between different dragging modalities takes place in Cabri is very important: abduction is linked to the perception of the effect of the movement of the mouse on the screen.

- Cabri is a mediator because of the dragging function, which allows students to perform experiments with geometrical figures, that otherwise would be difficult because left only to the mind. Dragging figures on the screen allows students to do such explorations as those the experts can spontaneously do in the resolution of a problem, in paper and pencil. So we can conclude that the different dragging modalities are crucial for producing a shift from conjecturing to proving. In this sense the world of the geometric figures in Cabri can be seen as a "field of experience" (Boero et al, 1996), in that it allows manipulation and 'experiences' of objects on the screen.

Further issues to be addressed are:

- Investigating the interaction between perception (of the movement in Cabri), anticipation (of what I want to do with the mouse) and logical relations (which translate the movement seen in Cabri into logical sentences 'if…then') in students' approaches to problems in Cabri.
- Moreover it would be interesting to study the same process of transition between the two modalities of control within other media used to approach geometry, as for example mathematical machines (Bartolini Bussi, 1993). Is it possible to adapt these analytical tools (control and abduction) in order to study the mental processes involved in goal oriented explorations of both physical and virtual instruments?
- Analysing in more depth what meanings students produce for what they see on the screen, how they interpret dragging and in which way they transform visual relationships between points in logical relationship between statements.

References

Arsac, G., Germain, G. & Mante, M. (1988) Probleme ouvert et situation-probleme, *IREM*, Academie de Lyon.

Arzarello, F., Andriano, V., Olivero, F. & Robutti, O. (1999). Abduction and conjecturing in mathematics, paper presented at the Symposium: *Model Based Reasoning in Scientific Discovery,* Pavia. In Magnani, L., Nersessian N.J. & Thagard, P. (eds.), Plenum Publishers.

Arzarello, F., Gallino, G., Micheletti, C., Olivero, F., Paola, D. & Robutti, O. (1998b) Dragging in Cabri and modalities of transition from conjectures to proofs in geometry, *Proceedings of PME XXII*, Stellenbosch, v.2, 32-39.

Arzarello, F., Micheletti, C., Olivero, F., Paola, D. & Robutti, O. (1998a) A model for analysing the transition to formal proofs in geometry, *Proceedings of PME XXII*, Stellenbosch, v.2, 24-31.

Arzarello, F., Olivero, F., Paola, D. & Robutti, O.: (1998c), Dalle congetture alle dimostrazioni. Una possibile continuità cognitiva., *L'insegnamento della matematica e delle scienze integrate*, vol.12B, n.3.

Balacheff, N. (1988) *Une Étude des processus de preuve en mathématique chez les élèves de collège*, Thèse d'état, Université Scientifique de Grenoble.

Barbin, E. (1988) La démostration mathématique: significations épistémologiques et questions didactiques, *Bulletin APMEP*, 366, 591-620.

Bartolini Bussi, M.G. (1993) Geometrical Proof and Mathematical Machines: An Exploratory Study, Proceedings of PME-XVII.

Boero, P., Garuti, R. & Mariotti, M.A. (1996) Some dynamic mental process underlying producing and proving conjectures, *Proceedings of PME XX*, Valencia, v. 2, 121-128.

Chazan, D. (1993) High school geometry student's justification for their views of empirical evidence and mathematical proof, *Educational Studies in Mathematics*, 24, 359-387.

Duval, R. (1991) Structure du raisonnement deductif et apprentissage de la démonstration, *Educational studies in mathematics*, 22, 233-261.

Duval, R. (1992) Argumenter, démontrer, expliquer: continuité ou rupture cognitive, *Petit X*, 31, 37-61.

Gallo, E. (1994) Control and solution of "algebraic problems". In: Arzarello F. & Gallo E. (Eds.), Problems in algebraic learning, special issue of *Rendiconti del Seminario matematico dell'Università e del Politecnico di Torino*, vol. 52, n.3, 263-278.

Goldenberg, E.P. (1995). Ruminations about dynamic imagery (and a strong plea for research). In: R. Sutherland & J. Mason, *Exploiting Mental Imagery with computers in mathematics education*, NATO Asi Series, Springer-Verlag.

Hanna, G. (1996) The ongoing value of proof, *Proceedings of PME XX*, Valencia, v. 1, 21-34.

Harel, G.& Sowder, L. (1996) Classifying processes of proving, *Proceedings of PME XX*, Valencia, v.3, 59-66.

Hölzl, R. (1995) Between drawing and figure. In: R. Sutherland and J. Mason (Eds.), *Exploiting Mental Imagery with Computers in Mathematical Education*, Berlin, Springer, 117-124.

Hölzl, R.: (1996) How does 'dragging' affect the learning of geometry, *Intern. Journ. of Computers for Mathematical Learning*, 1, 169-187.

Laborde, C. & Laborde, J.M.: (1992) Problem Solving in Geometry: From Microworlds to Intelligent Computer Environments. In: Mathematical Problem Solving and New Information Technologies. Research in Contexts of Practice, *NATO ASI Series*, Series F, 89.

Laborde, C. (1993) The computer as part of the learning environment: The case of geometry. In: C. Keitel and K. Ruthven (Eds.), *Learning from Computers: Mathematics Education and Technology*, Berlin, Springer, 48-67.

Laborde, C (1995) Designing tasks for learning geometry in computer-based environments. In: L. Burton & B. Jaworski (eds) *Technology in mathematics teaching: a bridge between teaching and learning*, Chartwell-Bratt, 35-68.

Magnani, L. (1997) Ingegnerie della conoscenza, Marcos y Marcos, Milano.

Mariotti, M.A., Bartolini Bussi, M., Boero, P., Ferri, F. & Garuti, R. (1997), Approaching geometry theorems in contexts: from history and epistemology to cognition, *Proceedings of PMEXXI*, Lathi, v.1, 180-195.

Moore, R.C. (1994) Making the transition to formal proof, *Educational Studies in Mathematics*, 27, 249-266.

Pea, R.D. (1987) Cognitive technologies for mathematics education. In: A.Schoenfeld (Ed.), *Cognitive Science and Mathematical Education*, Hillsdale, N.J., LEA Publ., 89-112.

Peirce, C.S. (1960) *Collected Papers*, II, Elements of Logic, Harvard, University Press, 372.

Polya, G. (1957) How to solve it: A new aspect of mathematical method, New York, Doubleday.

Saada-Robert, M. (1989) *La microgénèse de la rapresentation d'un problème*, Psychologie Française, 34, 2/3.

Simon, M. (1996) Beyond Inductive and Deductive Reasoning: the Search for a Sense of Knowing, *Educational Studies in mathematics*, 30, 197-210.

Thurston, W.P. (1995) On proof and progress in mathematics, *For the learning of mathematics*, 15, 1, 29-37.
Vygotskij, L.S. (1992) Pensiero e linguaggio, ed. critica di L. Mecacci, Laterza, Bari.